

# UNBOUNDED TWISTS OF HOOLEY'S $\Delta$ -FUNCTION

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**ABSTRACT.** We prove tight estimates for averages of the twisted Hooley  $\Delta$ -function over arbitrary number fields.

## 1. INTRODUCTION

In his memoir [Hoo79] Hooley studied the following function, previously brought to attention by Erdős,

$$\Delta(n) := \sup_{a \in \mathbb{R}} \sum_{\substack{d|n \\ e^a < d \leq e^{a+1}}} 1, \quad (n \in \mathbb{N}). \quad (1.1)$$

He showed that its average order is appreciably smaller than that of the divisor function, namely

$$\frac{1}{x} \sum_{n \leq x} \Delta(n) \ll (\log x)^{\frac{4}{\pi}-1}.$$

This saving enabled Hooley to provide new results in areas as diverse as Diophantine approximation, divisor sums and problems of Waring's type. Further applications were later found by Vaughan [Vau85], [Vau86], for problems of Waring's type, by Tenenbaum [Ten86] in the topic of Diophantine approximation, as well as for Chebychev's problem on the greatest prime factor of polynomial sequences by Tenenbaum [Ten90].

The problem regarding the average of  $\Delta$  was later revisited by Tenenbaum [Ten85], who managed to establish a strong upper bound, with a special corollary that the exponent  $\frac{4}{\pi} - 1$  can be replaced by any positive constant. Specifically, letting

$$\widehat{\varepsilon}(x) = \sqrt{\frac{\log \log \log(16 + x)}{\log \log(3 + x)}},$$

for any  $x \geq 1$ , enables us to state his result, namely

$$\frac{1}{x} \sum_{n \leq x} \Delta(n) \ll (\log x)^{O(\widehat{\varepsilon}(x))}.$$

Our first task in this paper is to generalise this to arbitrary number fields. Let  $K$  be any number field with ring of integers denoted by  $\mathcal{O}_K$ . The symbol  $\mathcal{I}_K$  will be reserved for the monoid of non-zero integral ideals of  $\mathcal{O}_K$ , while  $\mathfrak{N}\mathfrak{a} = \# \mathcal{O}_K / \mathfrak{a}$  will always refer to the ideal norm of  $\mathfrak{a} \in \mathcal{I}_K$ . The generalisation of (1.1) to  $K$  is given by

$$\Delta_K(\mathfrak{a}) := \sup_{a \in \mathbb{R}} \sum_{\substack{\mathfrak{d}|\mathfrak{a} \\ e^a < \mathfrak{N}\mathfrak{d} \leq e^{a+1}}} 1, \quad (\mathfrak{a} \in \mathcal{I}_K).$$

**Theorem 1.1.** *There exists a positive constant  $c = c(K)$  such that*

$$\frac{1}{x} \sum_{\mathfrak{N}\mathfrak{a} \leq x} \Delta_K(\mathfrak{a}) \ll (\log x)^{c\hat{\varepsilon}(x)},$$

where the implied constant is allowed to depend on  $K$ .

Introducing the arithmetic function,

$$r_K(d) := \#\{\mathfrak{d} \in \mathcal{I}_K : \mathfrak{N}\mathfrak{d} = d\}, \quad (d \in \mathbb{N}),$$

we see that

$$\Delta_K(\mathfrak{a}) = \sup_{a \in \mathbb{R}} \sum_{\substack{d|\mathfrak{N}\mathfrak{a} \\ e^a < d \leq e^{a+1}}} r_K(d).$$

Therefore  $\Delta_K$  can be perceived as a variant of  $\Delta$  over  $\mathbb{Z}$  twisted by  $r_K$ . The twist exhibits less tame behaviour than the constant function 1 or a Dirichlet character, as it may often assume *arbitrarily large values*. It is therefore perhaps surprising that the strong bounds valid for the classical Hooley  $\Delta$ -function can be extended to this setting. The proof of Theorem 1.1, given in §3, is based on the method involving the use of differential inequalities in [HT88, §7].

Next, let  $\psi_K$  be any quadratic Dirichlet character on  $K$  and define

$$\Delta_K(\mathfrak{a}; \psi_K) := \sup_{\substack{a \in \mathbb{R} \\ 0 \leq b \leq 1}} \left| \sum_{\substack{\mathfrak{d}|\mathfrak{a} \\ e^a < \mathfrak{N}\mathfrak{d} \leq e^{a+b}}} \psi_K(\mathfrak{d}) \right|, \quad (\mathfrak{a} \in \mathcal{I}_K). \quad (1.2)$$

In the special case  $K = \mathbb{Q}$  this function was considered by Daniel (see [Brü12]), as well as de la Bretèche and Tenenbaum [dlBT12]. Their work culminates in the bound,

$$\frac{1}{x} \sum_{1 \leq n \leq x} \Delta_{\mathbb{Q}}(n; \psi_{\mathbb{Q}})^2 \ll (\log x)^{O(\hat{\varepsilon}(x))}. \quad (1.3)$$

Our second undertaking in the present work is to generalise this to any number field.

**Theorem 1.2.** *Let  $\psi_K$  be a quadratic Dirichlet character defined on any number field  $K$ . There exists a positive constant  $c = c(K, \psi_K)$  such that*

$$\frac{1}{x} \sum_{\mathfrak{N}\mathfrak{a} \leq x} \Delta_K(\mathfrak{a}; \psi_K)^2 \ll (\log x)^{c\hat{\varepsilon}(x)},$$

where the implied constant is allowed to depend on  $K$  and  $\psi_K$ .

Our method, when fully extracted, is capable of proving estimates for higher moments. In particular, it is able of showing that the average of  $\Delta_K(\mathfrak{a})^2$  is  $\ll (\log \mathfrak{N}\mathfrak{a})^{1+o(1)}$  and, as remarked after the statement of Theorem 3 in [HT86], the exponent of the logarithm is expected to be best possible up to  $o(1)$ . Thus an important feature of Theorem 1.2 is that it provides a *square-root saving* over the estimate that one would obtain by bounding  $\Delta_K(\mathfrak{a}; \psi_K)$  by  $\Delta_K(\mathfrak{a})$ .

The proof of Theorem 1.2, supplied in §4, makes use of the approach in [dlBT12] involving an induction related to the number of prime ideal divisors of  $\mathfrak{a}$ . There will be a slight modification however; this is to take care of the fact that there may be several ideals  $\mathfrak{d}$  of a given norm present in (1.2). Therefore very short sums  $\sum_{\mathfrak{d}} \psi_K(\mathfrak{d})$  may contain an unusually large amount of terms and one is required to show that cancellation among the values  $\psi_K(\mathfrak{d})$  still occurs. This situation does not arise when  $K = \mathbb{Q}$ .

Let us finally remark that interest in averages of  $\Delta$ -functions has lately spiked due to applications to Manin's conjecture. This is a central conjecture in the area of Diophantine geometry, introduced by Manin and his collaborators in [FMT89], whose aim is to provide a precise description of the distribution of rational points on Fano varieties. However, its status for surfaces has not yet been fully resolved. An important rôle in proving the conjecture for Châtelet surfaces is assumed by the asymptotic estimation of divisor sums of the form

$$\sum_{\substack{(s,t) \in \mathbb{Z}^2 \\ |s|, |t| \leq x}} \sum_{\substack{d \in \mathbb{N} \\ d|F(s,t)}} \psi(d),$$

as  $x \rightarrow +\infty$ , where  $F \in \mathbb{Z}[s, t]$  is a separable quartic form and  $\psi$  is a quadratic Dirichlet character. De la Bretèche and Tenenbaum used (1.3) to handle these divisor sums when  $F$  is irreducible or a product of two irreducible quadratic forms in [dlBT13], which enabled them to prove Manin's conjecture for two families of Châtelet surfaces.

In our forthcoming joint work [BS16] with Browning, Theorems 1.1 and 1.2 are used to study divisor sums of the shape

$$\sum_{\substack{(s,t) \in \mathbb{Z}^2 \\ |s|, |t| \leq x}} \left( \prod_{i=1}^n \sum_{\substack{d_i | F_i(s,t) \\ d_i \text{ odd}}} \left( \frac{G_i(s,t)}{d_i} \right) \right),$$

where  $F_i, G_i \in \mathbb{Z}[s, t]$  are appropriate binary forms with  $\sum_{i=1}^n \deg(F_i) = 4$  and  $(\frac{\cdot}{\cdot})$  denotes the Jacobi symbol. As a byproduct we provide matching upper and lower bounds agreeing with Manin's conjecture for every quartic del Pezzo surface with a conic bundle structure over  $\mathbb{Q}$ .

**Notation.** The symbol  $\mathfrak{p}$  will exclusively refer throughout this paper to prime ideals in  $\mathcal{O}_K$  and the residue degree of any  $\mathfrak{p} \subset \mathcal{O}_K$  will be denoted by  $f_{\mathfrak{p}}$ . We shall make frequent use of the multiplicative span of all linear prime ideals,

$$\mathcal{P}_K = \{\mathfrak{a} \subset \mathcal{O}_K : \mathfrak{p} \mid \mathfrak{a} \Rightarrow f_{\mathfrak{p}} = 1\}.$$

The symbols  $\mu_K, \tau_K$  and  $\Lambda_K$  will be used for the Möbius, divisor and the von Mangoldt function on  $\mathcal{I}_K$ , while  $\omega_K$  will stand for the number of distinct prime ideal divisors on  $\mathcal{I}_K$ . Unless the contrary is explicitly stated, the implicit constants in Landau's  $O$ -notation and Vinogradov's  $\ll$ -notation are allowed to depend on  $K$  and  $\psi_K$  but no other parameters. Lastly, the notation  $f(x) \asymp g(x)$  will be taken to mean  $f(x) \ll g(x) \ll f(x)$ .

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## 2. PRECURSORY MANEUVERS

We begin by establishing the following property,

$$\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K \text{ coprime} \Rightarrow \Delta_K(\mathfrak{a}\mathfrak{b}) \leq \tau_K(\mathfrak{a})\Delta_K(\mathfrak{b}) \text{ and } \Delta_K(\mathfrak{a}\mathfrak{b}; \psi_K) \leq \tau_K(\mathfrak{a})\Delta_K(\mathfrak{b}; \psi_K). \quad (2.1)$$

Indeed, any  $\mathfrak{d} \mid \mathfrak{a}\mathfrak{b}$  can be written uniquely as  $\mathfrak{d} = \mathfrak{d}_1\mathfrak{d}_2$ , where  $\mathfrak{d}_1 \mid \mathfrak{a}$ ,  $\mathfrak{d}_2 \mid \mathfrak{b}$ . Therefore

$$\sum_{\substack{\mathfrak{d} \mid \mathfrak{a}\mathfrak{b} \\ e^a < \mathfrak{N}\mathfrak{d} \leq e^{a+b}}} \psi_K(\mathfrak{d}) = \sum_{\mathfrak{d}_1 \mid \mathfrak{a}} \psi_K(\mathfrak{d}_1) \sum_{\substack{\mathfrak{d}_2 \mid \mathfrak{b} \\ e^{a-\log \mathfrak{N}\mathfrak{d}_1} < \mathfrak{N}\mathfrak{d}_2 \leq e^{a+b-\log \mathfrak{N}\mathfrak{d}_1}}} \psi_K(\mathfrak{d}_2)$$

and a similar equality holds when  $\psi_K$  is replaced by 1. By the triangle inequality we ensure the validity of (2.1).

**Lemma 2.1.** *For any  $W_0 \in \mathbb{N}$  and any  $f : \mathcal{S}_K \rightarrow \mathbb{R}_{\geq 0}$  define the pair of functions*

$$M(x; f) := 1 + \sup_{1 \leq y \leq x} \frac{1}{y} \sum_{\mathfrak{N}\mathfrak{a} \leq y} f(\mathfrak{a})$$

and

$$L(x, W_0; f) := 1 + \sup_{1 \leq y \leq x} \frac{1}{\log y} \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq y \\ \mathfrak{a} \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{a}, W_0) = 1}} \frac{f(\mathfrak{a}) \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}}.$$

*If there exists  $t > 0$  such that  $f(\mathfrak{a}\mathfrak{b}) \leq \tau_K(\mathfrak{a})^t f(\mathfrak{b})$  for all integral coprime ideals  $\mathfrak{a}, \mathfrak{b}$  then for any  $W_0 \in \mathbb{N}$  we have the following as  $x \rightarrow \infty$ ,*

$$M(x; f) \asymp_{t, W_0} L(x, W_0; f).$$

*Proof.* Let us begin by showing that

$$\sum_{\mathfrak{N}\mathfrak{a} \leq x} \frac{f(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \asymp_{W_0} \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x \\ \mathfrak{a} \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{a}, W_0) = 1}} \frac{f(\mathfrak{a}) \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}}. \quad (2.2)$$

The non-negativity of  $f$  makes the inequality  $\gg$  clear. To prove the remaining inequality we may factorise uniquely each  $\mathfrak{a} \in \mathcal{S}_K$  as  $\mathfrak{a} = \mathfrak{b}\mathfrak{c}\mathfrak{d}$ , where each prime ideal divisor  $\mathfrak{p}$  of  $\mathfrak{b}$  satisfies  $\mathfrak{N}\mathfrak{p} | W_0$  and each prime ideal factor of  $\mathfrak{a}$  which is coprime to  $W_0$  and has residue degree at least 2 divides  $\mathfrak{c}$ . The property of  $f$  stated in our lemma shows that

$$\sum_{\mathfrak{N}\mathfrak{a} \leq x} \frac{f(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \leq \prod_{\mathfrak{N}\mathfrak{p} | W_0} \left( \sum_{m=0}^{\infty} \frac{\tau_K(\mathfrak{p}^m)^t}{\mathfrak{N}\mathfrak{p}^m} \right) \prod_{\substack{\mathfrak{N}\mathfrak{p} \leq x \\ \mathfrak{p} \nmid W_0}} \left( \sum_{m=0}^{\infty} \frac{\tau_K(\mathfrak{p}^m)^t}{\mathfrak{N}\mathfrak{p}^m} \right) \sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x \\ \mathfrak{d} \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{d}, W_0) = 1}} \frac{f(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}}$$

and we see that the first term is  $O_{t, W_0}(1)$ . Writing  $\mathfrak{N}\mathfrak{p} = p^g$  for a rational prime  $p$  we see that the second product is

$$\ll \prod_{2 \leq g \leq [K:\mathbb{Q}]} \prod_{p \leq x^{1/g}} \left( \sum_{m=0}^{\infty} \frac{(m+1)^t}{p^{gm}} \right) \leq \prod_{p \leq x} \left( 1 + O_t\left(\frac{1}{p^2}\right) \right)^{[K:\mathbb{Q}]} \ll 1.$$

It thus remains to show that

$$\sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x \\ \mathfrak{d} \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{d}, W_0) = 1}} \frac{f(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} \ll \sum_{\substack{\mathfrak{N}\mathfrak{d}_1 \leq x \\ \mathfrak{d}_1 \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{d}_1, W_0) = 1}} \frac{f(\mathfrak{d}_1) \mu_K(\mathfrak{d}_1)^2}{\mathfrak{N}\mathfrak{d}_1}.$$

To this end, we may factorise uniquely each  $\mathfrak{d}$  as  $\mathfrak{d}_1 \mathfrak{d}_2$  where  $\mathfrak{d}_1, \mathfrak{d}_2$  are coprime,  $\mathfrak{d}_1$  is square-free and  $\mathfrak{d}_2$  is square-full. We may thus infer that

$$\sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x \\ \mathfrak{d} \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{d}, W_0) = 1}} \frac{f(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} \leq \sum_{\substack{\mathfrak{N}\mathfrak{d}_1 \leq x \\ \mathfrak{d}_1 \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{d}_1, W_0) = 1}} \frac{f(\mathfrak{d}_1) \mu_K(\mathfrak{d}_1)^2}{\mathfrak{N}\mathfrak{d}_1} \sum_{\substack{\mathfrak{N}\mathfrak{d}_2 \leq x \\ \mathfrak{p} | \mathfrak{d}_2 \Rightarrow \mathfrak{p}^2 | \mathfrak{d}_2}} \frac{\tau_K(\mathfrak{d}_2)^t}{\mathfrak{N}\mathfrak{d}_2}$$

and the proof of (2.2) is concluded by observing that the sum over  $\mathfrak{d}_2$  is

$$\leq \prod_{\mathfrak{N}\mathfrak{p} \leq x} \left( 1 + O_t\left(\frac{1}{\mathfrak{N}\mathfrak{p}^2}\right) \right) = O_t(1).$$

In light of (2.2) it is sufficient for our lemma to show that

$$M(x; f) \asymp_t 1 + \sup_{1 \leq y \leq x} \frac{1}{\log y} \sum_{\mathfrak{N}\mathfrak{a} \leq y} \frac{f(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}. \quad (2.3)$$

Abel's summation can be employed to prove the inequality  $\gg$  in (2.3). For the remaining inequality let us factorise  $\mathfrak{a}$  as  $\mathfrak{b}\mathfrak{c}$  with  $\mathfrak{b}, \mathfrak{c}$  coprime,  $\mathfrak{b}$  square-free and  $\mathfrak{c}$  square-full. This yields

$$\sum_{\mathfrak{N}\mathfrak{a} \leq y} f(\mathfrak{a}) \leq \sum_{\substack{\mathfrak{N}\mathfrak{c} \leq y \\ \mathfrak{p}|\mathfrak{c} \Rightarrow \mathfrak{p}^2|\mathfrak{c}}} \tau_K(\mathfrak{c})^t \sum_{\mathfrak{N}\mathfrak{b} \leq y/\mathfrak{N}\mathfrak{c}} f(\mathfrak{b}) \mu_K(\mathfrak{b})^2.$$

Therefore, if the following holds

$$\sum_{\mathfrak{N}\mathfrak{b} \leq T} f(\mathfrak{b}) \mu_K(\mathfrak{b})^2 \ll T^{\frac{1}{2}} + \frac{T}{\log T} \sum_{\mathfrak{N}\mathfrak{b} \leq T} \frac{f(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}}, \quad (2.4)$$

then the required estimate (2.3) becomes available thanks to

$$\begin{aligned} \sum_{\mathfrak{N}\mathfrak{a} \leq y} f(\mathfrak{a}) &\ll y \sum_{\substack{\mathfrak{N}\mathfrak{c} \leq y \\ \mathfrak{p}|\mathfrak{c} \Rightarrow \mathfrak{p}^2|\mathfrak{c}}} \frac{\tau_K(\mathfrak{c})^t}{\mathfrak{N}\mathfrak{c}} \frac{1}{\log \frac{x}{\mathfrak{N}\mathfrak{c}}} \sum_{\mathfrak{N}\mathfrak{b} \leq x/\mathfrak{N}\mathfrak{c}} \frac{f(\mathfrak{b}) \mu_K(\mathfrak{b})^2}{\mathfrak{N}\mathfrak{b}} \\ &\ll y \left( \sup_{1 \leq y \leq x} \frac{1}{\log y} \sum_{\mathfrak{N}\mathfrak{b} \leq y} \frac{f(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}} \right) \sum_{\substack{\mathfrak{N}\mathfrak{c} \leq y \\ \mathfrak{p}|\mathfrak{c} \Rightarrow \mathfrak{p}^2|\mathfrak{c}}} \frac{\tau_K(\mathfrak{c})^t}{\mathfrak{N}\mathfrak{c}} \end{aligned}$$

and

$$\sum_{\substack{\mathfrak{N}\mathfrak{c} \leq y \\ \mathfrak{p}|\mathfrak{c} \Rightarrow \mathfrak{p}^2|\mathfrak{c}}} \frac{\tau_K(\mathfrak{c})^t}{\mathfrak{N}\mathfrak{c}} \ll \prod_{\mathfrak{N}\mathfrak{p} \leq y} \left( 1 + O_t \left( \frac{1}{\mathfrak{N}\mathfrak{p}^2} \right) \right) \ll_t 1.$$

To prove (2.4) we shall deploy the bound  $f(\mathfrak{b}) \leq \tau_K(\mathfrak{b}) \ll \mathfrak{N}\mathfrak{b}^{1/2}$  to obtain

$$\begin{aligned} \sum_{\mathfrak{N}\mathfrak{b} \leq T} f(\mathfrak{b}) \mu_K(\mathfrak{b})^2 &= \sum_{\mathfrak{N}\mathfrak{b} \leq T^{1/4}} f(\mathfrak{b}) \mu_K(\mathfrak{b})^2 + \sum_{T^{1/4} < \mathfrak{N}\mathfrak{b} \leq T} f(\mathfrak{b}) \mu_K(\mathfrak{b})^2 \\ &\ll T^{1/2} + \sum_{\mathfrak{N}\mathfrak{b} \leq T} f(\mathfrak{b}) \mu_K(\mathfrak{b})^2 \frac{\log \mathfrak{N}\mathfrak{b}}{\log T}. \end{aligned}$$

Employing the identity  $\mu_K(\mathfrak{b})^2 \log \mathfrak{N}\mathfrak{b} = \sum_{\mathfrak{b}=\mathfrak{c}\mathfrak{p}} \log \mathfrak{N}\mathfrak{p}$  allows us to bound the last sum by

$$\frac{2}{\log T} \sum_{\mathfrak{N}\mathfrak{c} \leq T} f(\mathfrak{c}) \mu_K(\mathfrak{c})^2 \sum_{\mathfrak{N}\mathfrak{p} \leq T/\mathfrak{N}\mathfrak{c}} \log \mathfrak{N}\mathfrak{p} \ll \frac{T}{\log T} \sum_{\mathfrak{N}\mathfrak{c} \leq T} \frac{f(\mathfrak{c})}{\mathfrak{N}\mathfrak{c}},$$

where the prime number theorem for  $K$  has been used.  $\square$

The purpose of Lemma 2.1 is twofold. Firstly, it allows the deduction of the estimates required in [BS16], the precise formulation of which is given below and differs from Theorems 1.1 and 1.2.

**Proposition 2.2.** *(i) There exists a positive constant  $c = c(K)$  such that*

$$\sum_{\substack{\mathfrak{a} \in \mathcal{P}_K \\ \mathfrak{N}\mathfrak{a} \leq x}} \frac{\Delta_K(\mathfrak{a}) \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll (\log x)^{1+c\widehat{\varepsilon}(x)}.$$

(ii) *There exists a positive constant  $c = c(K, \psi_K)$  such that for any  $W \in \mathbb{N}$  we have*

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x, \mathfrak{a} \in \mathcal{P}_K \\ \gcd(\mathfrak{N}\mathfrak{a}, W) = 1}} \frac{\Delta_K(\mathfrak{a}; \psi_K)^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll (\log x)^{1+c\widehat{\varepsilon}(x)}.$$

*The implied constant in both estimates is allowed to depend on  $K$  and, in the second estimate, also on  $W$  and the character  $\psi_K$ .*

*Proof.* By (2.1) we are allowed to deploy Lemma 2.1 with  $f = \Delta_K$ ,  $t = 1$  and  $W_0 = 1$  to obtain the estimate of the first part by Theorem 1.1. The second part stems similarly from Theorem 1.2 by taking  $f(\mathfrak{a}) = \Delta_K(\mathfrak{a}; \psi_K)^2$ ,  $t = 2$  and  $W_0 = W$  in Lemma 2.1.  $\square$

The second application of Lemma 2.1 is that it makes possible to deduce Theorems 1.1 and 1.2 from the following claims. *There exist a positive constants  $c_1, z_1$  that depend only on  $K$  such that*

$$\sum_{\mathfrak{N}\mathfrak{a} \leq x} \frac{\Delta_K(\mathfrak{a}) \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll_K (\log x)^{1+c_1\widehat{\varepsilon}(x)}. \quad (2.5)$$

*Furthermore, for any Dirichlet quadratic character  $\psi_K$  there exist positive constants  $c_2, z_2$  that depend only on  $K$  and  $\psi_K$  such that*

$$\sum_{\substack{\mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \mathfrak{a} \in \mathcal{P}_K, \mathfrak{N}\mathfrak{a} \leq x}} \frac{\Delta(\mathfrak{a}; \psi_K)^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll_{K, \psi_K} (\log x)^{1+c_2\widehat{\varepsilon}(x)}. \quad (2.6)$$

### 3. THE PROOF OF THEOREM 1.1

The Dirichlet series  $\sum_{\mathfrak{b} \in \mathcal{O}_K} \tau_K(\mathfrak{b})^2 \mathfrak{N}\mathfrak{b}^{-\sigma} = \zeta_K(\sigma)^2$  has a double pole at  $\sigma = 1$ . Thus there exists  $\alpha_K > 0$  such that for all  $\sigma \in (1, 2]$ ,

$$\sum_{\mathfrak{b} \in \mathcal{O}_K} \frac{\tau_K(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}^\sigma} \leq \frac{\alpha_K}{(\sigma - 1)^2},$$

and, similarly,  $-\frac{\zeta'_K(\sigma)}{\zeta_K(\sigma)}$  has a pole of order 1 at  $\sigma = 1$ , hence for some  $\gamma_K > 0$  we have for all  $\sigma \in (1, 2]$  that

$$\sum_{\mathfrak{p}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^\sigma} \leq \frac{\gamma_K}{\sigma - 1}.$$

We shall find it convenient to deploy the following constant throughout this section,

$$\beta_K := \sup_{z \in \mathbb{R}} \sum_{z \leq \mathfrak{N}\mathfrak{p} < ez} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}}.$$

Clearly, we may assume that

$$\min\{\alpha_K, \beta_K, \gamma_K\} > e, \quad (3.1)$$

upon replacing each constant by a larger value if needed. For  $\mathfrak{a} \in \mathcal{O}_K$ ,  $u \in \mathbb{R}$  and  $q \in \mathbb{R}_{\geq 1}$  we let

$$\Delta_K(\mathfrak{a}; u) := \sum_{\substack{\mathfrak{d}|\mathfrak{a} \\ e^u < \mathfrak{N}\mathfrak{d} \leq e^{u+1}}} 1 \quad \text{and} \quad M_q(\mathfrak{a}) := \int_{-\infty}^{+\infty} \Delta_K(\mathfrak{a}; u)^q du. \quad (3.2)$$

Let us begin by remarking that the proof of the inequality

$$\Delta_K(\mathfrak{a})^q \leq 2^{q-1} M_q(\mathfrak{a}), \quad (q \in \mathbb{N}, \mathfrak{a} \subset \mathcal{O}_K), \quad (3.3)$$

can be achieved in an identical manner as in [HT88, Th. 72] and will therefore not be given here.

**Lemma 3.1.** *For all  $\mathfrak{a} \in \mathcal{I}_K$  and  $q \in \mathbb{N}$  we have  $M_q(\mathfrak{a}) \leq \tau_K(\mathfrak{a})^q$ .*

*Proof.* It is evident that  $M_q(\mathfrak{a}) \leq \Delta_K(\mathfrak{a}) M_{q-1}(\mathfrak{a})$ , hence the assertion can be validated by induction on  $q$  upon noting that  $M_1(\mathfrak{a}) = \tau_K(\mathfrak{a})$ .  $\square$

Let us now bring into play the function

$$L(\sigma) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{M_q(\mathfrak{a})^{\frac{1}{q}} \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}^\sigma}, \quad (q \in \mathbb{N}_{\geq 2}, \sigma \in (1, 2]).$$

The equality  $\mu_K(\mathfrak{a})^2 \log \mathfrak{N}\mathfrak{a} = \mu_K(\mathfrak{a})^2 \sum_{\mathfrak{a}=\mathfrak{p}\mathfrak{b}} \log \mathfrak{N}\mathfrak{p}$  makes evident that

$$-L'(\sigma) \leq \sum_{\mathfrak{b} \subset \mathcal{O}_K} \frac{\mu_K(\mathfrak{b})^2}{\mathfrak{N}\mathfrak{b}^\sigma} \sum_{\mathfrak{p} \nmid \mathfrak{b}} \frac{M_q(\mathfrak{p}\mathfrak{b})^{\frac{1}{q}}}{\mathfrak{N}\mathfrak{p}^\sigma} \log \mathfrak{N}\mathfrak{p}. \quad (3.4)$$

By Hölder's inequality with exponents  $q$  and  $\frac{q}{q-1}$  the sum over  $\mathfrak{p}$  is

$$\leq \left( \sum_{\mathfrak{p}} M_q(\mathfrak{p}\mathfrak{b}) \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^\sigma} \right)^{\frac{1}{q}} \left( \sum_{\mathfrak{p}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^\sigma} \right)^{\frac{q-1}{q}}.$$

Let us set for  $a_i \in \mathbb{R}_{\geq 0}, w \in \mathbb{R}$ ,

$$N(\mathfrak{a}; w; a_0, a_1) := \int_{-\infty}^{+\infty} \Delta(\mathfrak{a}; u)^{a_0} \Delta(\mathfrak{a}; u - w)^{a_1} du.$$

To deal with  $M_q(\mathfrak{p}\mathfrak{b})$  when  $\mathfrak{p} \nmid \mathfrak{b}$  we use the identity

$$\Delta(\mathfrak{p}\mathfrak{b}; u) = \Delta(\mathfrak{b}; u - \log \mathfrak{N}\mathfrak{p}) + \Delta(\mathfrak{b}; u)$$

to arrive at

$$M_q(\mathfrak{p}\mathfrak{b}) = \sum_{a=0}^q \binom{q}{a} N(\mathfrak{b}; \log \mathfrak{N}\mathfrak{p}; q - a, a). \quad (3.5)$$

Let us record here the evident equalities

$$N(\mathfrak{a}; \log \mathfrak{N}\mathfrak{p}; 0, q) = N(\mathfrak{a}; \log \mathfrak{N}\mathfrak{p}; q, 0) = M_q(\mathfrak{b}). \quad (3.6)$$

**Lemma 3.2.** *For each  $\mathfrak{b} \subset \mathcal{O}_K$  and integers  $a, q$  with  $1 \leq a \leq q - 1$  we have*

$$\sum_{\mathfrak{p}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \Delta(\mathfrak{b}; u - \log \mathfrak{N}\mathfrak{p})^a \leq \beta_K \sum_{\substack{\mathfrak{d}_1, \dots, \mathfrak{d}_a \\ \mathfrak{d}_i \mid \mathfrak{b} \\ \max \mathfrak{N}\mathfrak{b}_i < e \min \mathfrak{N}\mathfrak{b}_i}} 1.$$

*Proof.* The sum in the lemma equals

$$\sum_{\substack{\mathfrak{d}_1, \dots, \mathfrak{d}_a \\ \mathfrak{d}_i \mid \mathfrak{b}}} \sum^* \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}},$$

where  $\sum^*$  is taken over  $\mathfrak{p}$  satisfying

$$u - \log \mathfrak{N}\mathfrak{p} < \min \log \mathfrak{N}\mathfrak{b}_i \leq \max \log \mathfrak{N}\mathfrak{b}_i \leq u - \log \mathfrak{N}\mathfrak{p} + 1.$$

In particular, we have

$$\max \log \mathfrak{N}\mathfrak{b}_i - \min \log \mathfrak{N}\mathfrak{b}_i < 1, u - \min \log \mathfrak{N}\mathfrak{b}_i < \log \mathfrak{N}\mathfrak{p} \leq u + 1 - \max \log \mathfrak{N}\mathfrak{b}_i$$

and  $\sum^* \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \leq \beta_K$ , which concludes our proof.  $\square$

**Lemma 3.3.** *For each  $\mathfrak{b} \subset \mathcal{O}_K$  and positive integer  $a$  we have*

$$\sum_{\substack{\mathfrak{d}_1, \dots, \mathfrak{d}_a \\ \mathfrak{d}_i | \mathfrak{b} \\ \max \mathfrak{N}\mathfrak{b}_i < e \min \mathfrak{N}\mathfrak{b}_i}} 1 \leq 2^{a+1} M_a(\mathfrak{b}).$$

*Proof.* It is convenient to rewrite the last summation condition as

$$2 - \log \left( \frac{\max \mathfrak{N}\mathfrak{b}_i}{\min \mathfrak{N}\mathfrak{b}_i} \right) > 1,$$

hence, letting  $x^+ = \max\{0, x\}$  for  $x \in \mathbb{R}$ , we can bound the sum in the lemma by

$$\sum_{\substack{\mathfrak{d}_1, \dots, \mathfrak{d}_a \\ \mathfrak{d}_i | \mathfrak{b}}} \left( 2 - \log \left( \frac{\max \mathfrak{N}\mathfrak{b}_i}{\min \mathfrak{N}\mathfrak{b}_i} \right) \right)^+.$$

Using the convention  $(a, b] = \emptyset$  when  $a \geq b$  verifies the succeeding identity for all  $a, b \in \mathbb{R}$ ,

$$(b - a)^+ = \int_{(a, b]} 1 du.$$

This provides the equality of the last sum with

$$\sum_{\substack{\mathfrak{d}_1, \dots, \mathfrak{d}_a \\ \mathfrak{d}_i | \mathfrak{b}}} \int_{[\log \max \mathfrak{N}\mathfrak{b}_i, 2 + \log \min \mathfrak{N}\mathfrak{b}_i)} 1 du = \int_{-\infty}^{+\infty} \left( \sum_{\substack{e^{u-2} < \mathfrak{N}\mathfrak{d} \leq e^u \\ \mathfrak{d} | \mathfrak{b}}} 1 \right)^a du,$$

which, upon decomposing the sum over  $\mathfrak{d}$  as

$$\sum_{\substack{e^{u-2} < \mathfrak{N}\mathfrak{d} \leq e^{u-1} \\ \mathfrak{d} | \mathfrak{b}}} 1 + \sum_{\substack{e^{u-1} < \mathfrak{N}\mathfrak{d} \leq e^u \\ \mathfrak{d} | \mathfrak{b}}} 1,$$

leads to the desired bound

$$2^a \int_{-\infty}^{+\infty} \left( \sum_{\substack{e^{u-2} < \mathfrak{N}\mathfrak{d} \leq e^{u-1} \\ \mathfrak{d} | \mathfrak{b}}} 1 \right)^a du + 2^a \int_{-\infty}^{+\infty} \left( \sum_{\substack{e^{u-1} < \mathfrak{N}\mathfrak{d} \leq e^u \\ \mathfrak{d} | \mathfrak{b}}} 1 \right)^a du,$$

that is clearly sufficient for the lemma.  $\square$

**Lemma 3.4.** *For each  $\mathfrak{b} \subset \mathcal{O}_K$  and integers  $a, q$  with  $1 \leq a \leq q - 1$  we have*

$$\sum_{\mathfrak{p}} N(\mathfrak{b}; \log \mathfrak{N}\mathfrak{p}; q - a, a) \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \leq \beta_K 2^q \tau_K(\mathfrak{b})^{\frac{q}{q-1}} M_q(\mathfrak{b})^{\frac{q-2}{q-1}}.$$



*Proof.* For integers  $c, q$  in the range  $1 \leq c \leq q-1$  we can obtain via Hölder's inequality with exponents  $\frac{q-1}{q-c}$  and  $\frac{q-1}{c-1}$  the succeeding inequality

$$M_c(\mathfrak{b}) = \int_{-\infty}^{+\infty} \Delta(\mathfrak{b}; u)^{\frac{q-c}{q-1}} \Delta(\mathfrak{b}; u)^{\frac{q(c-1)}{q-1}} du \leq M_1(\mathfrak{b})^{\frac{q-c}{q-1}} M_q(\mathfrak{b})^{\frac{c-1}{q-1}}.$$

Using this for  $c = a$  and  $c = q - a$  yields respectively

$$M_a(\mathfrak{b}) \leq M_1(\mathfrak{b})^{\frac{q-a}{q-1}} M_q(\mathfrak{b})^{\frac{a-1}{q-1}} \quad \text{and} \quad M_{q-a}(\mathfrak{b}) \leq M_1(\mathfrak{b})^{\frac{a}{q-1}} M_q(\mathfrak{b})^{\frac{q-a-1}{q-1}}. \quad (3.7)$$

Taking into account that  $M_1(\mathfrak{b}) = \tau_K(\mathfrak{b})$  and combining the two inequalities implies that

$$M_a(\mathfrak{b}) M_{q-a}(\mathfrak{b}) \leq \tau_K(\mathfrak{b})^{\frac{q}{q-1}} M_q(\mathfrak{b})^{\frac{q-2}{q-1}}.$$

Finally, bringing together Lemmas 3.2 and 3.3 shows that for each  $1 \leq a \leq q-1$  we have

$$\sum_{\mathfrak{p}} N(\mathfrak{b}; \log \mathfrak{N}\mathfrak{p}; q - a, a) \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \leq \beta_K 2^q M_a(\mathfrak{b}) M_{q-a}(\mathfrak{b}),$$

which makes the lemma evident.  $\square$

**Lemma 3.5.** *For each  $\sigma \in (1, 2]$  and integer  $q$  we have*

$$-L'(\sigma) \leq \gamma_K 2^{\frac{1}{q}} \frac{L(\sigma)}{\sigma - 1} + 4\beta_K^{\frac{1}{q}} \gamma_K^{\frac{q-1}{q}} \alpha_K^{\frac{2}{q-1}} L(\sigma)^{\frac{q-2}{q-1}} (\sigma - 1)^{-\frac{q^2+1}{q^2-q}}.$$

*Proof.* Let us begin by invoking (3.5), (3.6) and Lemma 3.4 to obtain

$$\sum_{\mathfrak{p}} M_q(\mathfrak{p}\mathfrak{b}) \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^\sigma} \leq 2\gamma_K \frac{M_q(\mathfrak{b})}{\sigma - 1} + 4^q \beta_K \tau_K(\mathfrak{b})^{\frac{q}{q-1}} M_q(\mathfrak{b})^{\frac{q-2}{q-1}}.$$

Observe that Hölder's inequality yields the estimate

$$\sum_{\mathfrak{p}} M_q(\mathfrak{p}\mathfrak{b})^{\frac{1}{q}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^\sigma} \leq \left( \sum_{\mathfrak{p}} M_q(\mathfrak{p}\mathfrak{b}) \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^\sigma} \right)^{\frac{1}{q}} \left( \frac{\gamma_K}{\sigma - 1} \right)^{\frac{q-1}{q}},$$

which, in combination with  $(a+b)^{1/q} \leq a^{1/q} + b^{1/q}$ , valid for each  $a, b \geq 0$ , yields

$$\sum_{\mathfrak{p}} M_q(\mathfrak{p}\mathfrak{b})^{\frac{1}{q}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^\sigma} \leq \frac{2^{\frac{1}{q}} \gamma_K M_q(\mathfrak{b})^{\frac{1}{q}}}{\sigma - 1} + \frac{4\beta_K^{\frac{1}{q}} \gamma_K^{\frac{q-1}{q}} \tau_K(\mathfrak{b})^{\frac{1}{q-1}} M_q(\mathfrak{b})^{\frac{q-2}{q(q-1)}}}{(\sigma - 1)^{\frac{q-1}{q}}},$$

that leads to the following estimate, once (3.4) has been taken into account,

$$-L'(\sigma) \leq \frac{\gamma_K 2^{\frac{1}{q}}}{\sigma - 1} L(\sigma) + 4\beta_K^{\frac{1}{q}} \left( \frac{\gamma_K}{\sigma - 1} \right)^{\frac{q-1}{q}} \sum_{\mathfrak{b}} \frac{\mu_K(\mathfrak{b})^2}{\mathfrak{N}\mathfrak{b}^\sigma} \tau_K(\mathfrak{b})^{\frac{1}{q-1}} M_q(\mathfrak{b})^{\frac{q-2}{q(q-1)}}.$$

We next employ Hölder's inequality with exponents  $q-1$  and  $\frac{q-1}{q-2}$  to bound the last sum by the quantity  $\alpha_K^{\frac{2}{q-1}} L(\sigma)^{\frac{q-2}{q-1}} (\sigma - 1)^{-\frac{2}{q-1}}$ , which validates the claim of the lemma.  $\square$

Assume that  $q$  is an even positive integer and define the constants

$$\gamma := 1 + \frac{1}{q} \quad \text{and} \quad g_0 := 4^{q-1} \beta_K^{\frac{q-1}{q}} \gamma_K^{\frac{(q-1)^2}{q}} \alpha_K^2 \left( \gamma_K 2^{1/q} - 1 - \frac{1}{q} \right)^{-q+1}.$$

Recalling (3.1), we see that the inequality

$$\left(\gamma_K 2^{1/q} - 1 - \frac{1}{q}\right)^{q-1} \leq 4^{q-1} \beta_K^{\frac{q-1}{q}} \gamma_K^{\frac{(q-1)^2}{q}} \alpha_K$$

holds, as long as  $q \gg_K 1$ . Let us now introduce the function  $X(\sigma) := g_0(\sigma - 1)^{-\gamma}$  in the domain  $1 < \sigma \leq 2$ . Note that Lemma 3.1 supplies us with  $L(2) \leq \alpha_K$ , hence the inequality above can be rewritten as  $L(2) < X(2)$ . Alluding to [HT88, Lemma 70.2] and making use of Lemma 3.5 provides us with  $L(\sigma) \leq X(\sigma)$  for each  $\sigma \in (1, 2]$ , hence the choice

$$q := 1 + 2[(\log \log x)^{1/2} (\log \log \log x)^{-1/2}],$$

yields a positive constant  $c_1 = c_1(K)$ , such that

$$\sum_{\mathfrak{N}\mathfrak{a} \leq x} \frac{M_q(\mathfrak{a})^{\frac{1}{q}} \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll_K L\left(1 + \frac{1}{\log x}\right) \ll_K 4^q (\log x)^{1+\frac{1}{q}} \ll (\log x)^{1+c_1 \widehat{\varepsilon}(x)}.$$

The required estimate (2.5) now flows directly upon employing (3.3).

#### 4. THE PROOF OF THEOREM 1.2

Recall that for the purpose of verifying Theorem 1.2 it is sufficient to prove the bound predicted by (2.6). We shall achieve this by an induction process which is brought into life in §4.1. The central result deployed in this process is Proposition 4.2, whose proof is postponed until §4.2.

**4.1. The induction process.** Throughout §4 the positive real number  $z_2 = z_2(K, \psi_K)$  will be allowed to increase but it will be independent of the counting parameter  $x$ . The Erdős–Kac theorem for  $K$  shows that the number of distinct prime ideal divisors of a typical element  $\mathfrak{a} \in \mathcal{J}_K$  is of size  $\log \log \mathfrak{N}\mathfrak{a}$ , thus suggesting to consider the contribution of  $\mathfrak{a}$  satisfying  $\omega_K(\mathfrak{a}) > 10 \log \log x$  in (2.6). Using (2.1) with  $\mathfrak{b} = \mathcal{O}_K$  we see that it is at most

$$\sum_{\substack{\omega_K(\mathfrak{a}) > 10 \log \log x \\ \mathfrak{N}\mathfrak{a} \leq x}} \frac{\tau_K(\mathfrak{a})^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \leq \sum_{\mathfrak{N}\mathfrak{a} \leq x} \frac{\tau_K(\mathfrak{a})^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \left(\frac{5}{2}\right)^{\omega_K(\mathfrak{b}) - 10 \log \log x}$$

and the inequality  $10 - 10 \log(\frac{5}{2}) < 1$  affirms the bound

$$(\log x)^{-10 \log(\frac{5}{2})} \prod_{\mathfrak{N}\mathfrak{p} \leq x} \left(1 + \frac{10}{\mathfrak{N}\mathfrak{p}}\right) \ll \log x.$$

This shows that (2.6) stems from the estimate

$$\sum_{\substack{\mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \omega_K(\mathfrak{a}) \leq 10 \log \log x \\ \mathfrak{a} \in \mathcal{J}_K, \mathfrak{N}\mathfrak{a} \leq x}} \frac{\Delta_K(\mathfrak{a}; \psi_K)^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll_{K, \psi_K} (\log x)^{1+c_2 \widehat{\varepsilon}(x)}. \quad (4.1)$$

We will soon replace the  $\Delta$ -term by an expression involving an integral that approximates  $\Delta_K(\mathfrak{a}; \psi_K)$ . The approximation can be performed when the divisors of  $\mathfrak{a}$  are evenly spaced and we proceed by showing that the sum in (4.1) can be restricted to  $\mathfrak{a}$  with this property. For any  $A > 0$  we define  $\mathcal{E}(A)$  as the set of all  $\mathfrak{a} \in \mathcal{J}_K$  for which there are distinct  $\mathfrak{d}, \mathfrak{d}'$  with

$$\mathfrak{d}|\mathfrak{a}, \mathfrak{d}'|\mathfrak{a}, \quad \mathfrak{N}\mathfrak{d} \leq \mathfrak{N}\mathfrak{d}' \leq \mathfrak{N}\mathfrak{d}(1 + (\log 2\mathfrak{N}\mathfrak{d})^{-A}).$$

Assume that  $A \geq 10$ . Then each ideal counted in

$$\sum_{\substack{\mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \mathfrak{a} \in \mathcal{P}_K \cap \mathcal{E}(A), \mathfrak{N}\mathfrak{a} \leq x}} \frac{\Delta_K(\mathfrak{a}; \psi_K)^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \quad (4.2)$$

is of the shape  $\mathfrak{a} = \mathfrak{d}\mathfrak{d}'\mathfrak{m}$ , where  $\mathfrak{d}, \mathfrak{d}', \mathfrak{m}$  are coprime in pairs and square-free and satisfy

$$\mathfrak{N}\mathfrak{d} \leq \mathfrak{N}\mathfrak{d}' \leq \mathfrak{N}\mathfrak{d}(1 + (\log 2\mathfrak{N}\mathfrak{d})^{-A}).$$

Hence, by (2.1) with  $\mathfrak{b} = \mathcal{O}_K$ , the sum is bounded by

$$\sum_{\substack{\mathfrak{m}, \mathfrak{d} \in \mathcal{P}_K \\ \mathfrak{N}\mathfrak{m}\mathfrak{N}\mathfrak{d} \leq x}} \frac{\mu_K(\mathfrak{m})^2 \mu_K(\mathfrak{d})^2 \tau_K(\mathfrak{m})^2}{\mathfrak{N}\mathfrak{m}\mathfrak{N}\mathfrak{d} \tau_K(\mathfrak{m})^{-2}} \sum_{\substack{\mathfrak{d}' \in \mathcal{P}_K \\ \mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \mathfrak{N}\mathfrak{d} \leq \mathfrak{N}\mathfrak{d}' \leq \mathfrak{N}\mathfrak{d}(1 + (\log 2\mathfrak{N}\mathfrak{d})^{-A})}} \frac{\mu_K(\mathfrak{d}')^2 \tau_K(\mathfrak{d}')^2}{\mathfrak{N}\mathfrak{d}'}$$

Introducing the following arithmetic function,

$$f(d) := \sum_{\substack{\mathfrak{d}' \in \mathcal{P}_K \\ \mathfrak{N}\mathfrak{d}' = d}} \mu_K(\mathfrak{d}')^2 \tau_K(\mathfrak{d}')^2,$$

allows us to bound the sum over  $\mathfrak{d}'$  by

$$\mathfrak{N}\mathfrak{d}^{-1} \sum_{\substack{p|d \Rightarrow p > z_2 \\ \mathfrak{N}\mathfrak{d} \leq d \leq \mathfrak{N}\mathfrak{d}(1 + (\log 2\mathfrak{N}\mathfrak{d})^{-A})}} f(d).$$

Using [Shi80, Th.1] shows that the last expression is bounded by  $\mathfrak{N}\mathfrak{d}^{-1}(\log \mathfrak{N}\mathfrak{d})^{-1-A}$  multiplied by

$$\ll_A \exp \left( \sum_{z_2 < p \leq z_2 + 2\mathfrak{N}\mathfrak{d}} f(p)/p \right) \ll \exp \left( \sum_{z_2 < \mathfrak{N}\mathfrak{p} \leq z_2 + 2\mathfrak{N}\mathfrak{d}} 4/\mathfrak{N}\mathfrak{p} \right) \ll (\log \mathfrak{N}\mathfrak{d})^4.$$

We have thus shown that the sum in (4.2) is

$$\ll \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \mathfrak{N}\mathfrak{m} \leq x}} \frac{\mu_K(\mathfrak{m})^2 \tau_K(\mathfrak{m})^2}{\mathfrak{N}\mathfrak{m}} \sum_{\substack{\mathfrak{d} \in \mathcal{P}_K \\ \mathfrak{N}\mathfrak{d} \leq x}} \frac{\mu_K(\mathfrak{d})^2 \tau_K(\mathfrak{d})^2}{\mathfrak{N}\mathfrak{d}(\log \mathfrak{N}\mathfrak{d})^{A-3}} \ll (\log x)^4 \sum_{\substack{\mathfrak{d} \in \mathcal{P}_K \\ \mathfrak{N}\mathfrak{d} \leq x}} \frac{\mu_K(\mathfrak{d})^2 \tau_K(\mathfrak{d})^2}{\mathfrak{N}\mathfrak{d}(\log \mathfrak{N}\mathfrak{d})^{A-3}}.$$

By Abel's summation the sum over  $\mathfrak{d}$  is  $\ll (\log x)^{7-A}$ , thus yielding

$$A \geq 10 \Rightarrow \sum_{\substack{\mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \mathfrak{a} \in \mathcal{P}_K \cap \mathcal{E}(A), \mathfrak{N}\mathfrak{a} \leq x}} \frac{\tau_K(\mathfrak{a})^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll_A (\log x)^{11-A},$$

which reveals that, owing to (4.1), the next estimate is sufficient for the proof of (2.6),

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x, \mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \omega_K(\mathfrak{a}) \leq 10 \log \log x \\ \mathfrak{a} \in \mathcal{P}_K, \mathfrak{a} \notin \mathcal{E}(10)}} \frac{\Delta_K(\mathfrak{a}; \psi_K)^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \ll_{K, \psi_K} (\log x)^{1+c_2 \widehat{\varepsilon}(x)}. \quad (4.3)$$

The induction process that will enable us to prove (4.3) requires that we are in possession of an ordering of the prime ideals  $\mathfrak{p} \subset \mathcal{O}_K$ ; thus we form the sequence  $(\mathfrak{p}_i)_{i=1}^\infty$  such that

$$i < j \Rightarrow \mathfrak{p}_i \neq \mathfrak{p}_j, \mathfrak{N}\mathfrak{p}_i \leq \mathfrak{N}\mathfrak{p}_j. \quad (4.4)$$

Prime ideals of the equal norm are allowed to be ordered arbitrarily, but their ordering is fixed once and for all. Hence, for any  $\mathfrak{a}$  we can set  $i^+(\mathfrak{a}) = \max\{i \in \mathbb{N} : \mathfrak{p}_i \mid \mathfrak{a}\}$  and define

$$\mathfrak{p}^+(\mathfrak{a}) := \mathfrak{p}_{i^+(\mathfrak{a})}.$$

Furthermore, for each  $r \in \mathbb{N}$  and square-free  $\mathfrak{a} \in \mathcal{J}_K$ , we let  $\mathfrak{a}_r := \mathfrak{a}$  if  $r \geq \omega_K(\mathfrak{a})$ . If  $r < \omega_K(\mathfrak{a})$  holds then we choose the first  $r$  prime ideal divisors of  $\mathfrak{a}$  according to the ordering above and let  $\mathfrak{a}_r$  be their product. Setting  $r_x := \lceil 10 \log \log x \rceil$  shows that the sum in (4.3) is

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x, \mathfrak{p} \mid \mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \omega_K(\mathfrak{a}) \leq 10 \log \log x \\ \mathfrak{a} \in \mathcal{P}_K, \mathfrak{a} \notin \mathcal{E}(10)}} \frac{\Delta_K(\mathfrak{a}_{r_x}; \psi_K)^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}} \leq \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x, \mathfrak{p} \mid \mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2 \\ \mathfrak{a} \in \mathcal{P}_K}} \frac{\Delta_K(\mathfrak{a}_{r_x}; \psi_K)^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}}.$$

Letting for any  $\mathfrak{a} \in \mathcal{J}_K$ ,  $a \in \mathbb{R}$  and  $b \in (0, 1]$ ,

$$\Delta_K(\mathfrak{a}; \psi; a, b) := \left| \sum_{\substack{e^a < \mathfrak{N}\mathfrak{d} \leq e^{a+b} \\ \mathfrak{d} \mid \mathfrak{a}}} \psi_K(\mathfrak{d}) \right|$$

sets the stage for the entrance of the important entity

$$M_q(\mathfrak{a}; \psi_K) := \int_0^1 \int_{\mathbb{R}} \Delta_K(\mathfrak{a}; \psi; a, b)^q da db, \quad (\mathfrak{a} \in \mathcal{J}_K, q \in \mathbb{N}). \quad (4.5)$$

Let  $q_x := \lceil \sqrt{r_x / (1 + \log r_x)} \rceil$  and define for  $r, q \in \mathbb{N}$  the average

$$\mathcal{L}(x) := 4^{\frac{r_x}{q_x}} \log x + \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x, \mathfrak{a} \in \mathcal{P}_K \\ \mathfrak{p} \mid \mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \frac{M_{2q_x}(\mathfrak{a}_{r_x}; \psi_K)^{\frac{1}{q_x}} \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}}.$$

The next lemma shows that Theorem 1.2 stems from

$$\mathcal{L}(x) \ll_{K, \psi_K} (\log x)^{1+c_2 \hat{\varepsilon}(x)}. \quad (4.6)$$

**Lemma 4.1.** *For all  $q \in \mathbb{N}$  and square-free  $\mathfrak{a} \in \mathcal{J}_K$  with  $\mathfrak{N}\mathfrak{a} \leq x$  and  $\mathfrak{a} \notin \mathcal{E}(10)$  we have*

$$\Delta_K(\mathfrak{a}; \psi_K)^2 \leq 8^2 + 2^{10} (\log x)^{\frac{20}{q}} M_{2q}(\mathfrak{a}; \psi_K)^{\frac{1}{q}}.$$

*Proof.* The lemma is valid if  $\Delta_K(\mathfrak{a}; \psi_K) < 8$ , we may therefore assume henceforth that the opposite holds. Note that the definition of  $\Delta_K(\mathfrak{a}; \psi_K)$  provides  $a_0 \in \mathbb{R}, b_0 \in [0, 1]$  such that

$$|\Delta_K(\mathfrak{a}; \psi_K; a_0, b_0)| \geq \frac{1}{2} \Delta_K(\mathfrak{a}; \psi_K). \quad (4.7)$$

We bring into play the box  $\mathfrak{B} \subset \mathbb{R}^2$  given by

$$\left(a_0, a_0 + \frac{1}{8(\log 2x)^{10}}\right) \times \left(b_0, b_0 + \frac{1}{8(\log 2x)^{10}}\right), \left(a_0 - \frac{1}{8(\log 2x)^{10}}, a_0\right) \times \left(b_0 - \frac{1}{8(\log 2x)^{10}}, b_0\right)$$

respectively according to whether  $b_0 < \frac{1}{2}$  or not. We choose to focus on the latter case; the former being treated similarly. For any  $(a, b) \in \mathfrak{B}$  we have

$$|\Delta_K(\mathfrak{a}; \psi_K; a, b) - \Delta_K(\mathfrak{a}; \psi_K; a_0, b_0)| \leq \sum_{\substack{\mathfrak{d} \mid \mathfrak{a} \\ e^a \leq \mathfrak{N}\mathfrak{d} \leq e^{a_0}}} 1 + \sum_{\substack{\mathfrak{d} \mid \mathfrak{a} \\ e^{a+b} \leq \mathfrak{N}\mathfrak{d} \leq e^{a_0+b_0}}} 1. \quad (4.8)$$

If the first sum has more than one term then there exist  $\mathfrak{d} \neq \mathfrak{d}' \in \mathcal{J}_K$  with  $\mathfrak{d}, \mathfrak{d}' \mid \mathfrak{a}$  and

$$e^{a_0 - 1/8(\log 2x)^{10}} \leq \mathfrak{N}\mathfrak{d} \leq \mathfrak{N}\mathfrak{d}' \leq e^{u_0},$$

thus leading via  $\mathfrak{N}\mathfrak{a} \leq x$  and  $e^z \leq 1 + 2z$  (valid in the range  $0 < z < 1$ ), to

$$\mathfrak{N}\mathfrak{d}' \leq \mathfrak{N}\mathfrak{d} e^{1/8(\log 2x)^{10}} \leq \mathfrak{N}\mathfrak{d} \left(1 + \frac{1}{4(\log 2x)^{10}}\right) \leq \mathfrak{N}\mathfrak{d} \left(1 + \frac{1}{(\log 2\mathfrak{N}\mathfrak{d})^{10}}\right),$$

which contradicts the assumption  $\mathfrak{a} \notin \mathcal{E}(10)$  of our lemma. A similar argument shows that the second sum in (4.8) also contains at most one term, therefore invoking  $\Delta_K(\mathfrak{a}; \psi_K) \geq 8$  and (4.7) provides us with

$$\Delta_K(\mathfrak{a}; \psi_K; a, b) \geq \frac{\Delta_K(\mathfrak{a}; \psi_K)}{2} - 2 \geq \frac{\Delta_K(\mathfrak{a}; \psi_K)}{4}.$$

This inequality immediately furnishes the required estimate by restricting the range of integration in (4.5) to  $\mathfrak{B}$ .  $\square$

For positive integers  $r, q$  and any  $\sigma \in (0, \frac{1}{4}]$  we define the functions

$$L_{r,q}^*(\sigma) := \frac{4^{\frac{r}{q}}}{\sigma} + \sum_{\substack{\mathfrak{a} \in \mathcal{P}_K \\ \mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \frac{M_{2q}(\mathfrak{a}_r; \psi_K)^{\frac{1}{q}} \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}^{1+\sigma}}$$

and for  $s \in \mathbb{R}_{\geq 1}$  we let

$$f(s) := \sqrt{\frac{s}{1 + \log s}}.$$

Noting that  $\mathcal{L}(x) \leq eL_{r_x, q_x}^*(1/\log x)$ , our aim now becomes to prove that for all sufficiently small  $\sigma > 0$  we have

$$r \gg 1, q = [f(r)] \Rightarrow L_{r,q}^*(\sigma) \ll \frac{e^{c_2 \sqrt{r \log r}}}{\sigma} \quad (4.9)$$

for some constant  $c_2 > 0$  depending at most on  $K$  and  $\psi_K$ . Clearly, this is sufficient for verifying (4.6).

The strategy for the proof of (4.9) is indirect and resembles a backwards induction process. First, note that if the variable  $r$  is replaced by any fixed integer constant  $t$ , then for any  $q$  we have  $L_{t',q}^* \ll_{t'} 1/\sigma$ . Indeed, using Lemma 3.1 in combination with the obvious bound  $M_{2q}(\mathfrak{a}; \psi_K) \leq M_{2q}(\mathfrak{a})$  furnishes

$$L_{t',q}^*(\sigma) - \frac{4^{\frac{t'}{q}}}{\sigma} \ll \sum_{\mathfrak{a} \in \mathcal{J}_K} \frac{\tau_K(\mathfrak{a}_{t'})^2 \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}^{1+\sigma}} \leq \sum_{\mathfrak{a} \in \mathcal{J}_K} \frac{2^{2t'}}{\mathfrak{N}\mathfrak{a}^{1+\sigma}} \ll_{t'} \zeta_K(1 + \sigma) \ll_t \frac{1}{\sigma}. \quad (4.10)$$

It will therefore be advantageous to bound  $L_{r,q}^*(\sigma)$  in terms of  $L_{r-1,q}^*(\sigma)$  for  $r$  and  $q$  in suitable ranges. To this end we shall deploy the succeeding lemma, whose proof is postponed until §4.2.

**Proposition 4.2.** *There exist positive constants  $c_3, t', z_2, \sigma_0$  that depend at most on  $K$  and  $\psi_K$ , such that for all integers  $t, m$  in the range*

$$t' \leq t \leq 10 \log \frac{1}{\sigma}, \quad m \leq \sqrt{t/(1 + \log t)} < m + 2,$$

and  $\sigma \in (0, \sigma_0)$  we have

$$L_{t,m}^*(\sigma) \leq e^{\frac{c_3}{m}} L_{t-1,m}^*(\sigma).$$

To deduct (4.6) from Proposition 4.2 define for each integer  $\ell$  the following set,

$$\mathcal{A}_\ell := \{n \in \mathbb{N} : \ell \leq f(n) < \ell + 1\},$$

which furnishes the following partition into disjoint sets

$$\mathbb{N} \cap [t', r] = \bigcup_{\ell \in \mathbb{N}} \mathcal{A}_\ell.$$

Let  $k := \min \mathcal{A}_\ell$ . It is easy to see that  $f(k + c\sqrt{k \log k}) > f(k) + 1$  holds for some large positive  $c$  independent of  $k$ , and therefore  $\sharp \mathcal{A}_\ell \leq c\sqrt{\ell \log \ell}$ . In addition, the definition of  $k$  shows that  $f(k-1) < \ell$  and therefore  $\sqrt{k \log k} \leq \ell \log \ell$ , hence  $\sharp \mathcal{A}_\ell \leq c_4 \ell \log \ell$  for some absolute constant  $c_4 > 0$ . Furthermore,  $\mathcal{A}_\ell$  will be empty unless  $\ell \leq q$ .

It is now time to reveal our backwards induction process. Whenever  $n \in \mathcal{A}_q$  we use Proposition 4.2 with  $t = n, m = q$  to reduce the value of  $n$  from  $r$  down to  $\min \mathcal{A}_q$ . This will come at a cost of  $\exp(\frac{c_3}{q} \sharp \mathcal{A}_q) \leq q^{c_3 c_4}$ . At the end of this section we shall prove that there exists a positive constant  $c_5 = c_5(K, \psi_K)$  such that for all  $\sigma > 0$  sufficiently small we have

$$t' \leq t \leq 10 \log \frac{1}{\sigma}, \quad m = [f(t)] \quad \Rightarrow \quad L_{t,m}^*(\sigma) \leq m^{c_5} L_{t,m-1}^*(\sigma). \quad (4.11)$$

When  $n$  reaches  $\min \mathcal{A}_q$  we will use (4.11) with  $t = \min \mathcal{A}_q$  and  $m = q$ . We subsequently iterate the process by using Proposition 4.2 with  $m = q-1$  and  $t = n$  for all  $n \in \mathcal{A}_{q-1}$ . We repeat this procedure going backwards until  $\ell$  is small enough so that  $t' \in \mathcal{A}_\ell$ . The total cost will be

$$\ll \prod_{\ell \leq q} \ell^{c_3 c_4 + c_5} \leq e^{c_2 \sqrt{r \log r}},$$

for some constant  $c_2 > 0$  that depends at most on  $K$  and  $\psi_K$ . At the end of this process we shall be left with  $L_{t',q}^*(\sigma)$  which can be estimated via (4.10), thus concluding the proof of (4.9).

*Proof of (4.11).* Let us begin by introducing the constants  $\eta_K := \min \left\{ 10^{-3}, 2^{-\frac{3}{[K:\mathbb{Q}]}} \right\}$  and  $u_t := \exp(70^t/t)$ . We shall make use of the set  $\mathcal{D}_t$  that consists of all square-free  $\mathfrak{a} \in \mathcal{J}_K$  for which there are distinct  $\mathfrak{d}, \mathfrak{d}'$  satisfying

$$\mathfrak{d}|\mathfrak{a}_t, \mathfrak{d}'|\mathfrak{a}_t, \quad \mathfrak{N}\mathfrak{d} \leq \mathfrak{N}\mathfrak{d}' \leq \mathfrak{N}\mathfrak{d}(1 + \eta_K^t).$$

For each such  $\mathfrak{a}$  we can choose and fix square-free and coprime in pairs  $\mathfrak{d}_{\mathfrak{a}_t}, \mathfrak{d}'_{\mathfrak{a}_t}, \mathfrak{m}_{\mathfrak{a}_t} \in \mathcal{J}_K$  with  $\mathfrak{a} = \mathfrak{d}_{\mathfrak{a}_t} \mathfrak{d}'_{\mathfrak{a}_t} \mathfrak{m}_{\mathfrak{a}_t}$  and  $\mathfrak{d}_{\mathfrak{a}_t}, \mathfrak{d}'_{\mathfrak{a}_t}$  being in the range designated above. We may now deploy the inequality  $\mu_K(\mathfrak{a})^2 M_{2m}(\mathfrak{a}_t; \psi_K) \leq \mu_K(\mathfrak{a})^2 \tau_K(\mathfrak{a}_t)^{2m} = 4^{tm}$  to infer that for large enough  $t \geq t'$  the contribution of  $\mathfrak{a}$  with  $\mathfrak{N}\mathfrak{d}_{\mathfrak{a}_t} \leq \exp(70^t)$  towards  $L_{t,m}^*(\sigma) - 4^{\frac{t}{m}}/\sigma$  is at most

$$4^t \sum_{\substack{\mathfrak{a} \in \mathcal{D}_t \\ \eta_K^{-t} \leq \mathfrak{N}\mathfrak{d}_{\mathfrak{a}_t} \leq \exp(70^t)}} \frac{\mu_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{1+\sigma}} \leq 4^t \sum_{\substack{\mathfrak{m} \in \mathcal{J}_K \\ \eta_K^{-t} \leq \mathfrak{N}\mathfrak{m} \leq \exp(70^t)}} \frac{1}{\mathfrak{N}\mathfrak{d}^{1+\sigma} \mathfrak{N}\mathfrak{m}^{1+\sigma}} \sum_{\substack{\mathfrak{d}' \in \mathcal{J}_K \\ \mathfrak{N}\mathfrak{d} \leq \mathfrak{N}\mathfrak{d}' \leq \mathfrak{N}\mathfrak{d}(1+\eta_K^t)}} \frac{1}{\mathfrak{N}\mathfrak{d}'^{1+\sigma}}.$$

The estimate  $\sum_{\mathfrak{N}\mathfrak{d}' \leq x} 1 = c_K x + O_K(x^{1-\frac{1}{[K:\mathbb{Q}]}})$  shows that the sum over  $\mathfrak{d}'$  is

$$\leq \mathfrak{N}\mathfrak{d}^{-1-\sigma} \left( c_K \eta_K^t \mathfrak{N}\mathfrak{d} + O_K(\mathfrak{N}\mathfrak{d}^{1-\frac{1}{[K:\mathbb{Q}]}}) \right),$$

which provides the following bound,

$$\ll \frac{4^t}{\sigma} \left( \eta_K^t \sum_{\mathfrak{N}\mathfrak{d} \leq \exp(70^t)} \frac{1}{\mathfrak{N}\mathfrak{d}} + \sum_{\mathfrak{N}\mathfrak{d} \geq \eta_K^{-t}} \frac{1}{\mathfrak{N}\mathfrak{d}^{1+[K:\mathbb{Q}]}} \right) \ll \frac{4^t}{\sigma} \left( \eta_K^t 70^t + \eta_K^{t[K:\mathbb{Q}]} \right) \ll \frac{1}{\sigma 2^t}.$$

Let us now focus on the contribution of  $\mathfrak{a} \in \mathcal{D}_t$  with  $\mathfrak{N}\mathfrak{d}_{\mathfrak{a}_t} > \exp(70^t)$ . The cardinality of the prime ideal divisors of  $\mathfrak{a}$  in the range  $\mathfrak{N}\mathfrak{p} \leq \exp(70^t/t)$ , henceforth denoted by  $\omega(\mathfrak{a}; t)$ , cannot exceed  $t$ , otherwise the first  $t$  prime ideals dividing  $\mathfrak{a}$  will have norm in that range, thus  $\mathfrak{N}\mathfrak{d}_{\mathfrak{a}_t} \leq \mathfrak{N}\mathfrak{a}_t \leq (\exp(70^t/t))^t$ , which is contradiction. In the case where  $\sigma > (32/9)t70^{-t}$  we see, upon using  $\zeta_K(1+\sigma) \ll \sigma^{-1}$ , that the contribution of the ideals  $\mathfrak{a}$  under consideration towards  $L_{t,m}^*(\sigma) - 4^{\frac{t}{m}}/\sigma$  is at most

$$4^t \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\mu_K(\mathfrak{a})^2 \mathfrak{N}\mathfrak{a}_t^{\sigma/2}}{\mathfrak{N}\mathfrak{a}^{1+\sigma} u_t^{t\sigma/2}} \ll 4^t \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}^{1+\sigma/2} u_t^{t\sigma/2}} \ll \frac{(7/10)^t}{\sigma}.$$

In the remaining case  $\sigma \leq (32/9)t70^{-t}$  we set  $v := 2/(\log 70)$  and bound the contribution by

$$4^t \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\mu_K(\mathfrak{a})^2 v^{\omega(\mathfrak{a}; t)-t}}{\mathfrak{N}\mathfrak{a}^{1+\sigma}} \ll \frac{4^t}{v^t \sigma} \prod_{\mathfrak{N}\mathfrak{p} \leq \exp\{\sqrt{u_t}\}} (1 + \mathfrak{N}\mathfrak{p}^{-1})^{v-1},$$

which is again  $\ll (7/10)^t/\sigma$ . Thus far we have shown that

$$L_{t,m}^*(\sigma) \ll \frac{4^{\frac{t}{m}} + (7/10)^t}{\sigma} + \sum_{\substack{\mathfrak{a} \in \mathcal{P}_K, \mathfrak{a} \notin \mathcal{D}_t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \frac{M_{2m}(\mathfrak{a}_t; \psi_K)^{\frac{1}{m}} \mu_K(\mathfrak{a})^2}{\mathfrak{N}\mathfrak{a}^{1+\sigma}}. \quad (4.12)$$

Taking  $z_2 > 2$  we see that each  $\mathfrak{a}$  in the sum has odd norm, thus each element of the set

$$S := \{(a, b) \in (-1, 0] \times (0, 1) : a + b \geq 0\}$$

satisfies  $\mathbb{Z} \cap (e^u, e^{a+b}] = \{1\}$  and therefore  $\Delta_K(\mathfrak{a}; \psi_K; a, b) = 1$ . Hence, for any  $q \in \mathbb{N}$  we have  $M_{2q}(\mathfrak{a}; \psi_K) \geq \text{vol}(S) = 1/2$ . We can now imitate the proof of Lemma 4.1, replacing  $1/(\log 2x)^{10}$  by  $\eta_K^t$ , to prove that for all  $q \in \mathbb{N}$  and square-free  $\mathfrak{a} \in \mathcal{S}_K$  with  $\mathfrak{a} \notin \mathcal{D}_t$  we have

$$\Delta_K(\mathfrak{a}_t; \psi_K)^2 \leq 8^2 + 2^{10} \eta_K^{-\frac{2t}{q}} M_{2q}(\mathfrak{a}; \psi_K)^{\frac{1}{q}} \leq 2^{11} \eta_K^{-\frac{2t}{q}} M_{2q}(\mathfrak{a}_t; \psi_K)^{\frac{1}{q}}.$$

Using this for  $q = m - 1$  in combination with

$$M_{2m}(\mathfrak{a}_t; \psi_K) \leq \Delta_K(\mathfrak{a}_t; \psi_K)^2 M_{2m-2}(\mathfrak{a}_t; \psi_K)$$

leads to

$$M_{2m}(\mathfrak{a}_t; \psi_K)^{\frac{1}{m}} \leq 2^{\frac{11}{m}} \eta_K^{-\frac{2t}{m(m-1)}} M_{2m-2}(\mathfrak{a}_t; \psi_K)^{\frac{1}{m-1}}.$$

The proof of (4.11) is concluded by injecting the last inequality into (4.12) and making use of  $m \gg f(t)$  to derive  $\eta_K^{-\frac{2t}{m(m-1)}} \leq m^{c_5}$  for some positive constant  $c_5$  that depends at most on  $K$  and  $\psi_K$ .  $\square$

**4.2. The proof of Proposition 4.2.** To relate  $L_{t,m}^*(\sigma)$  and  $L_{t-1,m}^*(\sigma)$  demands that we have an understanding of the fluctuation of  $M_{2m}(\mathfrak{a}; \psi_K)^{\frac{1}{m}}$  as the number of prime ideal divisors of  $\mathfrak{a}$  varies. To this end, we observe that for any  $\mathfrak{a} \in \mathcal{J}_K$  and prime  $\mathfrak{p}$  we have

$$\Delta_K(\mathfrak{ap}; \psi_K; a, b) = \Delta_K(\mathfrak{a}; \psi_K; a, b) + \psi_K(\mathfrak{p}) \Delta_K(\mathfrak{a}, \psi_K; a - \log \mathfrak{N}\mathfrak{p}, b).$$

For a positive integer  $m$  we can raise to the power  $2m$  to obtain

$$\Delta_K(\mathfrak{ap}; \psi_K; a, b)^{2m} = \sum_{0 \leq j \leq 2m} \binom{2m}{j} \psi(\mathfrak{p})_K^{2m-j} \Delta_K(\mathfrak{a}; \psi_K; a, b)^j \Delta_K(\mathfrak{ap}; \psi_K; a - \log \mathfrak{N}\mathfrak{p}, b)^{q-j}.$$

Hence, letting for  $\mathfrak{a} \in \mathcal{J}_K$ ,  $w \in \mathbb{R}$ ,  $m \in \mathbb{N}$  and  $0 \leq j \leq m$ ,

$$N_{j,m}(\mathfrak{a}, w) := \int_0^1 \int_{\mathbb{R}} \Delta_K(\mathfrak{a}; \psi_K; a, b)^j \Delta_W(\mathfrak{a}, \psi; a - w, b)^{q-j} da db$$

and recalling (4.5) we arrive at

$$M_{2m}(\mathfrak{ap}; \psi_K) = 2M_{2m}(\mathfrak{a}; \psi_K) + \sum_{1 \leq j \leq 2m-1} \binom{2m}{j} \psi_K(\mathfrak{p})^j N_{j,2m}(\mathfrak{a}, \log \mathfrak{N}\mathfrak{p}).$$

If  $1 < j < m-1$  we use  $cd \leq \frac{c^2}{2} + \frac{d^2}{2}$  for

$$c = \Delta_K(\mathfrak{a}; \psi_K; a, b)^{j+1} \Delta_K(\mathfrak{a}; \psi_K; a - w, b)^{m-j-1}, d = \Delta_K(\mathfrak{a}; \psi; a, b)^j \Delta_K(\mathfrak{a}; \psi_K; a - w, b)^{m-j}$$

to acquire

$$N_{2j+1,2m}(\mathfrak{a}, w) \leq \frac{1}{2} N_{2j+2,2m}(\mathfrak{a}, w) + \frac{1}{2} N_{2j,2q}(\mathfrak{a}, w)$$

and the inequality  $cd \leq \frac{mc^2}{2} + \frac{d^2}{2m}$  yields in like manner

$$\begin{aligned} N_{1,2m}(\mathfrak{a}, w) &\leq \frac{m}{2} N_{2,2m}(\mathfrak{a}, w) + \frac{1}{2m} M_{2m}(\mathfrak{a}; \psi_K), \\ N_{2m-1,2m}(\mathfrak{a}, w) &\leq \frac{m}{2} N_{2m-2,2m}(\mathfrak{a}, w) + \frac{1}{2m} M_{2m}(\mathfrak{a}; \psi_K). \end{aligned}$$

Putting everything together, we have

$$M_{2m}(\mathfrak{ap}; \psi_K) \leq 4M_{2m}(\mathfrak{a}; \psi_K) + W_{2m}(\mathfrak{a}, \mathfrak{p}),$$

where

$$W_{2m}(\mathfrak{a}, \mathfrak{p}) := \sum_{1 \leq j \leq m-1} b_j \binom{2m}{2j} N_{2j,2m}(\mathfrak{a}, \log \mathfrak{N}\mathfrak{p}) \quad (4.13)$$

and the sequence given through

$$b_j := \begin{cases} 1 + \frac{m}{2m-1} + \frac{2}{3}(m-2) & \text{if } j = 1 \\ 1 + \frac{m}{2m-1} & \text{if } j = m-1 \\ 1 + \frac{j}{2m-2j-1} + \frac{m-j}{2j+1} & \text{otherwise.} \end{cases}$$

satisfies  $b_j \leq 1 + \frac{2}{3}m$ .

Assume that we are given  $\mathfrak{a} \in \mathcal{J}_K$  with  $\omega_K(\mathfrak{a}) > t-1$ . Then letting  $\mathfrak{p}_t(\mathfrak{a})$  be the  $t$ -th prime ideal factor of  $\mathfrak{a}$  according to the ordering (4.4) and using  $(y_1 + y_2)^{\frac{1}{m}} \leq y_1^{\frac{1}{m}} + y_2^{\frac{1}{m}}$ , valid for  $y_i \in \mathbb{R}_{\geq 0}$ , we deduce

$$M_{2m}(\mathfrak{a}_t; \psi_K)^{\frac{1}{m}} \leq 4^{\frac{1}{m}} M_{2m}(\mathfrak{a}_{t-1}; \psi_K)^{\frac{1}{m}} + W_{2m}(\mathfrak{a}_{t-1}, \mathfrak{p}_t(\mathfrak{a}))^{\frac{1}{m}}.$$



This inequality is also valid if  $\omega_K(\mathfrak{a}) \leq t-1$ , since in that case we have  $\mathfrak{a}_t = \mathfrak{a}_{t-1}$ . We obtain

$$L_{t,m}^*(\sigma) \leq 4^{\frac{1}{m}} L_{t-1,m}^*(\sigma) + \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m})=t-1 \\ \mathfrak{p}|\mathfrak{m} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \sum_{\substack{\mathfrak{p}_j \in \mathcal{P}_K \\ j > i^+(\mathfrak{m})}} W_{2m}(\mathfrak{m}, \mathfrak{p}_j)^{\frac{1}{m}} \sum_{\mathfrak{n}_t = \mathfrak{m}\mathfrak{p}_j} \frac{\mu_K^2(\mathfrak{n})}{\mathfrak{N}\mathfrak{n}^{1+\sigma}}.$$

Each ideal  $\mathfrak{n}$  is of the form  $\mathfrak{m}\mathfrak{p}_j\mathfrak{d}$ , where  $\mathfrak{d}$  is square-free and each prime divisor of  $\mathfrak{d}$ ,  $\mathfrak{p}_i|\mathfrak{d}$  satisfies  $i > j$ . We can therefore deduce that the sum over  $\mathfrak{n}$  is

$$\ll \sum_{\mathfrak{n}_t = \mathfrak{m}\mathfrak{p}_j} \frac{\mu_K^2(\mathfrak{n})}{\mathfrak{N}\mathfrak{n}^{1+\sigma}} \ll \mathfrak{N}\mathfrak{m}\mathfrak{p}_j^{-1-\sigma} \prod_{\mathfrak{N}\mathfrak{p} > \mathfrak{N}\mathfrak{p}_j} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}^{1+\sigma}}\right),$$

and, recalling that we denote the Dedekind zeta function of  $K$  by  $\zeta_K$ , we deduce that the last product is

$$\leq \zeta_K(1+\sigma) \prod_{\mathfrak{N}\mathfrak{p} \leq \mathfrak{N}\mathfrak{p}_j} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}^{1+\sigma}}\right)^{-1} \ll \frac{1}{\sigma} \prod_{\mathfrak{N}\mathfrak{p} \leq \mathfrak{N}\mathfrak{p}_j} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}^{1+\sigma}}\right).$$

The inequality  $\mathfrak{N}\mathfrak{p}^{-\sigma} \geq 1 - \sigma \log \mathfrak{N}\mathfrak{p}$  and Mertens' theorem show that the inner product is

$$\ll \prod_{\mathfrak{N}\mathfrak{p} \leq \mathfrak{N}\mathfrak{p}_j} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right) \exp\left(\sigma \sum_{\mathfrak{N}\mathfrak{p} \leq \mathfrak{N}\mathfrak{p}_j} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}}\right) \ll \frac{\mathfrak{N}\mathfrak{p}_j^\sigma}{\log \mathfrak{N}\mathfrak{p}_j},$$

thus showing that the sum over  $\mathfrak{n}$  is  $\ll_K \sigma^{-1} \mathfrak{N}\mathfrak{m}^{-1-\sigma} (\mathfrak{N}\mathfrak{p}_j \log \mathfrak{N}\mathfrak{p}_j)^{-1}$ . Letting for  $\mathfrak{m} \in \mathcal{P}_K$ ,

$$\mathcal{A}_m(\mathfrak{m}) := \sum_{\substack{\mathfrak{p}_j \in \mathcal{P}_K \\ j > i^+(\mathfrak{m})}} \frac{W_{2m}(\mathfrak{m}, \mathfrak{p}_j)^{\frac{1}{m}}}{\mathfrak{N}\mathfrak{p}_j \log \mathfrak{N}\mathfrak{p}_j},$$

we have thus obtained

$$L_{t,m}^*(\sigma) - 4^{\frac{1}{m}} L_{t-1,m}^*(\sigma) \ll \frac{1}{\sigma} \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m})=t-1 \\ \mathfrak{p}|\mathfrak{m} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \frac{\mu_K^2(\mathfrak{m})}{\mathfrak{N}\mathfrak{m}^{1+\sigma}} \mathcal{A}_m(\mathfrak{m}). \quad (4.14)$$

Using Hölder's inequality with exponents  $m, \frac{m}{m-1}$  we see that  $\mathcal{A}_m(\mathfrak{m})$  is at most

$$\left( \sum_{\substack{\mathfrak{p}_j \in \mathcal{P}_K \\ j > i^+(\mathfrak{m})}} \frac{W_{2m}(\mathfrak{m}, \mathfrak{p}_j) \log \mathfrak{N}\mathfrak{p}_j}{\mathfrak{N}\mathfrak{p}_j} \right)^{\frac{1}{m}} \left( \sum_{\substack{\mathfrak{p}_j \in \mathcal{P}_K \\ j > i^+(\mathfrak{m})}} \frac{1}{\mathfrak{N}\mathfrak{p}_j (\log \mathfrak{N}\mathfrak{p}_j)^{\frac{m+1}{m-1}}} \right)^{\frac{m-1}{m}}.$$

By the prime number theorem for  $K$  and partial summation we infer that with  $z := \mathfrak{N}\mathfrak{p}^+(\mathfrak{m})$  the last sum is at most

$$\leq \sum_{\mathfrak{N}\mathfrak{p} > z/3} \frac{1}{\mathfrak{N}\mathfrak{p} (\log \mathfrak{N}\mathfrak{p})^{\frac{m+1}{m-1}}} \ll (\log z)^{-\frac{m+1}{m-1}}$$

thus acquiring the validity of

$$\mathcal{A}_m(\mathfrak{m}) \ll \left( \sum_{\substack{\mathfrak{p}_j \in \mathcal{P}_K \\ j > i^+(\mathfrak{m})}} \frac{W_{2m}(\mathfrak{m}, \mathfrak{p}_j) \log \mathfrak{N}\mathfrak{p}_j}{\mathfrak{N}\mathfrak{p}_j} \right)^{\frac{1}{m}} (\log z)^{-\frac{m+1}{m-1}}. \quad (4.15)$$

For  $\vartheta \in \mathbb{R}$  and  $\mathfrak{a} \in \mathcal{J}_K$  define

$$\tau_K^*(\mathfrak{a}; \psi_K; \vartheta) := \sum_{\mathfrak{d}|\mathfrak{a}} \psi_K(\mathfrak{d}) \mathfrak{N}\mathfrak{d}^{i\vartheta} \quad \text{and} \quad \tau_K^*(\mathfrak{a}; \psi_K) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|\tau_K^*(\mathfrak{a}; \psi_K; \vartheta)|^2}{6 + \vartheta^2} d\vartheta. \quad (4.16)$$

**Lemma 4.3.** *For all  $\mathfrak{a} \in \mathcal{J}_K$  we have  $M_2(\mathfrak{a}; \psi_K) \leq \tau_K^*(\mathfrak{a}; \psi_K)$ .*

*Proof.* We start by using the following well-known formula, valid for all  $u, v, x \in \mathbb{R}$ ,

$$-\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{it(x-v)} - e^{it(x-u)}}{t} dt = \begin{cases} 1 & \text{if } u < x \text{ and } x < v, \\ 0 & \text{if } x < u \text{ or } x > v, \end{cases}$$

a proof of which can be found, for example, in [Wie33, §5], . The substitution  $t \mapsto 2\pi r$  gives

$$\Delta_K(\mathfrak{a}; \psi_K; a, b) = \int_{-\infty}^{+\infty} \left( \frac{1 - e^{-2\pi i r b}}{2\pi i r} \tau_K^*(\mathfrak{a}, \psi_K; 2\pi r) \right) e^{-2\pi i a r} dr$$

except when  $a, b$  assume a finite set of values, thus Plancherel's theorem leads to

$$\int_{-\infty}^{+\infty} \Delta_K(\mathfrak{a}; \psi_K; a, b)^2 da = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} \frac{1 - \cos(2\pi r b)}{r^2} |\tau_K^*(\mathfrak{a}, \psi_K; 2\pi r)|^2 dr.$$

It can then be inferred from  $\int_0^1 (1 - \cos(2\pi r b)) db = 1 - \frac{\sin(2\pi r)}{2\pi r}$  that  $2\pi^2 M_2(\mathfrak{a}; \psi_K)$  equals

$$\int_{-\infty}^{+\infty} \left( 1 - \frac{\sin(2\pi r)}{2\pi r} \right) \frac{|\tau_K^*(\mathfrak{a}, \psi_K; 2\pi r)|^2}{r^2} dr$$

and the inequality  $1 - \frac{\sin(2\pi r)}{2\pi r} \leq \frac{4\pi^2 r^2}{3+2\pi^2 r^2}$  furnishes the proof of our lemma.  $\square$

Define the arithmetic function  $g : \mathbb{N} \rightarrow \mathbb{Z}$  though  $g(n) := \#\{\mathfrak{p} \subset \mathcal{O}_K : \mathfrak{N}\mathfrak{p} = n\}$  and note that the prime number theorem for  $K$  provides a positive constant  $\varkappa$  such that

$$\sum_{1 \leq n \leq T} g(n) = \text{li}(T) + O(Te^{-(\log T)^\varkappa}). \quad (4.17)$$

Recall the definition of  $M_q(\mathfrak{a})$  in (3.2).

**Lemma 4.4.** *For all  $\Xi \geq 1$  and  $m \in \mathbb{N}$  we have*

$$\begin{aligned} m^{-1} 4^{-m} \sum_{\mathfrak{N}\mathfrak{p} > \Xi} \frac{W_{2m}(\mathfrak{m}, \mathfrak{p}) \log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} &\ll M_{2m}(\mathfrak{m}, \psi_K)^{\frac{m-2}{m-1}} \tau_K^*(\mathfrak{m}; \psi_K)^{\frac{m}{m-1}} \\ &+ e^{-(\log \Xi)^\varkappa} 4^m M_{2m}(\mathfrak{m})^{\frac{2m-2}{2m-1}} \tau_K(\mathfrak{m})^{\frac{2m}{2m-1}}. \end{aligned}$$

*Proof.* Using (4.13) shows that the sum in our lemma is bounded by

$$\left(1 + \frac{2}{3}m\right) \sum_{1 \leq j \leq m-1} \binom{2m}{2j} \sum_{\mathfrak{N}\mathfrak{p} > \Xi} \frac{N_{2j, 2m}(\mathfrak{m}, \log \mathfrak{N}\mathfrak{p}) \log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \quad (4.18)$$

and the inner sum can be recast as

$$\int_0^1 \int_{\mathbb{R}} \Delta_K(\mathfrak{m}; \psi_K; a, b)^{2j} \left( \sum_{\mathfrak{N}\mathfrak{p} > \Xi} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \Delta_K(\mathfrak{m}; \psi_K; a - \log \mathfrak{N}\mathfrak{p}, b)^{2m-2j} \right) da db.$$

Letting  $h := 2m - 2j$  allows us to see that the sum over  $\mathfrak{p}$  equals

$$\sum_{\substack{\mathfrak{d}_1, \dots, \mathfrak{d}_h \in \mathcal{J}_K \\ \mathfrak{d}_i | \mathfrak{m}}} \psi_K(\mathfrak{d}_1 \cdots \mathfrak{d}_h) \sum_{n > \Xi}^* g(n) \frac{\log n}{n},$$

where the sum  $\sum^*$  is over integers  $n$  satisfying the further condition

$$a - \min \log \mathfrak{N}\mathfrak{d}_i < \log n \leq a - \max \log \mathfrak{N}\mathfrak{d}_i + b.$$

This implies that the sum contains no terms unless  $\max \mathfrak{N}\mathfrak{d}_i < e^b \min \mathfrak{N}\mathfrak{d}_i$ , in which case (4.17) and Abel's summation provide the bound

$$\ll \int_{a - \min \log \mathfrak{N}\mathfrak{d}_i}^{a + b - \log \max \mathfrak{N}\mathfrak{d}_i} \mathbf{1}_{(0, \infty)}(t - \log \Xi) dt + e^{-(\log \Xi)^\kappa},$$

where  $\mathbf{1}_{(0, \infty)}$  denotes the characteristic function of the positive real numbers. This confirms

$$\sum_{\mathfrak{N}\mathfrak{p} > \Xi} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \Delta_K(\mathfrak{m}; \psi_K; a - \log \mathfrak{N}\mathfrak{p}, b)^h \ll \int_{\mathbb{R}} \Delta_K(\mathfrak{m}; \psi_K; s, b)^h ds + e^{-(\log \Xi)^\kappa} \sum_{\substack{\mathfrak{d}_1, \dots, \mathfrak{d}_h \in \mathcal{S}_K, \mathfrak{d}_i | \mathfrak{m} \\ \max \mathfrak{N}\mathfrak{d}_i < e \min \mathfrak{N}\mathfrak{d}_i}} 1$$

and according to Lemma 3.3 the inner sum is  $\ll 2^h M_h(\mathfrak{m})$ . Therefore the sum over the prime ideals in (4.18) is

$$\begin{aligned} &\ll \int_0^1 \int_{\mathbb{R}} \Delta_K(\mathfrak{m}; \psi_K; a, b)^{2j} da \int_{\mathbb{R}} \Delta_K(\mathfrak{m}; \psi_K; s, b)^{2m-2j} ds db \\ &+ e^{-(\log \Xi)^\kappa} 4^{m-j} M_{2j}(\mathfrak{m}) M_{2m-2j}(\mathfrak{m}), \end{aligned}$$

where we have used  $|\psi_K(\mathfrak{d})| \leq 1$  to dispense with the integration over  $0 \leq b \leq 1$  in the second term. In virtue of (3.7) and Lemma (4.3) one can show by following the argument involving Hölder's inequality at the end of the proof of [dlBT12, Lem. 2.4] that the last expression is

$$\ll M_{2m}(\mathfrak{m}; \psi_K)^{\frac{m-2}{m-1}} \tau^*(\mathfrak{m}; \psi_K)^{\frac{m}{m-1}} + e^{-(\log \Xi)^\kappa} 4^m M_{2m}(\mathfrak{m})^{\frac{2m-2}{2m-1}} \tau_K(\mathfrak{m})^{\frac{2m}{2m-1}},$$

which, in view of  $\sum_{1 \leq j \leq m-1} \binom{2m}{2j} = 4^m - 2$ , finishes our proof.  $\square$

We may now deploy the bound supplied by Lemma 4.4 in conjunction with (4.15) to obtain

$$\mathcal{A}_m(\mathfrak{m}) \ll \mathcal{B}_m(\mathfrak{m}) + \mathcal{C}_m(\mathfrak{m}),$$

where

$$\mathcal{B}_m(\mathfrak{m}) := \frac{M_{2m}(\mathfrak{m}; \psi_K)^{\frac{m-2}{m(m-1)}} \tau_K^*(\mathfrak{m}; \psi_K)^{\frac{1}{m-1}}}{(\log z)^{\frac{m+1}{m-1}}}$$

and

$$\mathcal{C}_m(\mathfrak{m}) := \frac{M_{2m}(\mathfrak{m})^{\frac{2m-2}{m(2m-1)}} \tau_K(\mathfrak{m})^{\frac{2}{2m-1}}}{e^{\frac{1}{m}(\log z)^\kappa}}.$$

Alluding to Lemma 3.1, the term  $\mathcal{C}_m(\mathfrak{m})$  makes the following contribution towards (4.14),

$$\ll \frac{1}{\sigma} \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m}) = t-1 \\ \mathfrak{p} | \mathfrak{m} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \frac{\mu_K^2(\mathfrak{m}) \tau_K(\mathfrak{m})^2}{\mathfrak{N}\mathfrak{m}^{1+\sigma}} \exp\left(-\frac{1}{m}(\log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}))^\kappa\right).$$

Each  $\mathfrak{m}$  above is the product of  $\mathfrak{p}^+(\mathfrak{m})$  and  $t-2$  prime ideals  $\mathfrak{p}_i$  with  $\mathfrak{N}\mathfrak{p}_i \leq \mathfrak{N}\mathfrak{p}^+(\mathfrak{m})$ . Taking into account the possible permutations of the ideals  $\mathfrak{p}_i$  shows that the sum over  $\mathfrak{m}$  is

$$\ll \frac{1}{(t-2)!} \sum_{\mathfrak{N}\mathfrak{p} > z_2} \frac{\exp\left(-\frac{1}{m}(\log \mathfrak{N}\mathfrak{p})^\kappa\right)}{\mathfrak{N}\mathfrak{p}} \left( \sum_{\mathfrak{N}\mathfrak{p}_i \leq \mathfrak{N}\mathfrak{p}} \frac{4}{\mathfrak{N}\mathfrak{p}_i} \right)^{t-2}.$$

The sum over  $\mathfrak{p}_i$  is at most  $4 \log_2 \mathfrak{N}\mathfrak{p} + O(1)$ , hence using the inequality  $\exp(-x) \leq \frac{\ell!}{x^\ell}$ , valid for all  $x \geq 0, \ell \in \mathbb{N}$ , we obtain that the expression above is bounded by

$$\ll \frac{4^t \ell! m^\ell}{(t-1)!} \sum_{\mathfrak{p} > z_2} \frac{1}{\mathfrak{N}\mathfrak{p}} \frac{(\log \log \mathfrak{N}\mathfrak{p})^{t-2}}{(\log \mathfrak{N}\mathfrak{p})^{\varkappa \ell}}.$$

We may suppose that  $t'$  satisfies  $(\frac{\varkappa t'}{5} - 1) > 1$  and  $t' > 5$ , so that upon choosing  $\ell := \lceil \frac{t'}{5} \rceil$  we see that the sum is

$$\ll \int_{z_2}^{\infty} \frac{(\log \log u)^t}{u(\log u)^{t/5}} du = \left( \frac{\varkappa t}{5} - 1 \right)^{-t} \int_{(\frac{\varkappa t}{5} - 1) \log \log z_2}^{\infty} \frac{v^t}{e^v} dv \leq \left( \frac{\varkappa t}{5} - 1 \right)^{-t} t!.$$

Therefore, using  $\log m = \frac{1}{2} \log t + O(\log \log t)$  (which is implied by the assumptions of Proposition 4.2), as well as  $\log n! = n \log n + O(n)$ , we see that the contribution of the entity  $\mathcal{C}_m(\mathfrak{m})$  towards (4.14) is

$$\ll \sigma^{-1} \frac{4^t \ell! m^\ell 10^t}{\lambda_0^\ell t^t} \leq \sigma^{-1} \exp\left(-\frac{7}{10} t \log t + O(t \log \log t)\right) \ll \frac{1}{\sigma(t!)^{2/3}}.$$

We now turn our attention to the contribution of  $\mathcal{B}_m(\mathfrak{m})$  to (4.14). It is at most

$$\sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m}) = t-1 \\ \mathfrak{p} | \mathfrak{m} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \frac{M_{2m}(\mathfrak{m}; \psi_K)^{\frac{m-2}{m(m-1)}} \tau_K^*(\mathfrak{m}; \psi_K)^{\frac{1}{m-1}}}{(\log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}))^{\frac{1}{m}}} \frac{\mu_K^2(\mathfrak{m})}{\mathfrak{N}\mathfrak{m}^{1+\sigma}} \frac{1}{\sigma \log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m})}. \quad (4.19)$$

For  $\mathfrak{m}$  as in the sum above we let  $S(\mathfrak{m})$  be the set of square-free elements  $\mathfrak{n} \in \mathcal{P}_K$  that are divisible by  $\mathfrak{m}$  with the further property that any prime ideal  $\mathfrak{p}_i | \mathfrak{n}$  with  $\mathfrak{p}_i \nmid \mathfrak{m}$  satisfies  $i > i^+(\mathfrak{m})$ . These ideals enjoy the property  $\mathfrak{n}_{t-1} = \mathfrak{m}$  and therefore

$$\sum_{\substack{\mathfrak{n} \in \mathcal{P}_K \\ \mathfrak{n}_{t-1} = \mathfrak{m}}} \frac{\mu_K(\mathfrak{n})^2}{\mathfrak{N}\mathfrak{n}^{1+\sigma}} \geq \sum_{\mathfrak{n} \in S(\mathfrak{m})} \frac{\mu_K(\mathfrak{n})^2}{\mathfrak{N}\mathfrak{n}^{1+\sigma}} \geq \frac{\mu_K(\mathfrak{m})^2}{\mathfrak{N}\mathfrak{m}^{1+\sigma}} \prod_{\substack{i > i^+(\mathfrak{m}) \\ \mathfrak{p}_i \in \mathcal{P}_K}} \left( 1 + \frac{1}{\mathfrak{N}\mathfrak{p}_i^{1+\sigma}} \right).$$

The effect of primes  $\mathfrak{p}_i$  with residue degree more than 1 is bounded by a constant depending only on  $K$ , thus the product is

$$\geq \zeta_K(1+\sigma) \prod_{i \leq i^+(\mathfrak{m})} \left( 1 - \frac{1}{\mathfrak{N}\mathfrak{p}_i^{1+\sigma}} \right) \geq \frac{1}{\sigma} \exp \left( - \sum_{i \leq i^+(\mathfrak{m})} \frac{1}{\mathfrak{N}\mathfrak{p}_i^{1+\sigma}} \right),$$

which by the Mertens theorem for  $K$  is

$$\geq \frac{1}{\sigma} \exp \left( - \sum_{i \leq i^+(\mathfrak{m})} \frac{1}{\mathfrak{N}\mathfrak{p}_i} \right) \geq \frac{1}{\sigma \log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m})}.$$

We deduce that the sum in (4.19) is

$$\ll \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m}) = t-1 \\ \mathfrak{p} | \mathfrak{m} \Rightarrow \mathfrak{N}\mathfrak{p} > z_2}} \frac{M_{2m}(\mathfrak{m}; \psi_K)^{\frac{m-2}{m(m-1)}} \tau_K^*(\mathfrak{m}; \psi_K)^{\frac{1}{m-1}}}{(\log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}))^{\frac{1}{m}}} \sum_{\substack{\mathfrak{n} \in \mathcal{P}_K \\ \mathfrak{n}_{t-1} = \mathfrak{m}}} \frac{\mu_K(\mathfrak{n})^2}{\mathfrak{N}\mathfrak{n}^{1+\sigma}}.$$

Observe that for each  $\mathbf{n}$  in the inner sum we have  $\omega_K(\mathbf{n}) \geq \omega_K(\mathbf{m}) = t - 1$  and therefore the double sum may be reshaped into

$$\sum_{\substack{\mathbf{n} \in \mathcal{P}_K \\ \omega_K(\mathbf{n}) \geq t-1}} \left( \frac{\mu_K(\mathbf{n})^2 M_{2m}(\mathbf{n}_{t-1}; \psi_K)^{\frac{m-2}{m(m-1)}}}{\mathfrak{N}\mathbf{n}^{\frac{(1+\sigma)(m-2)}{m-1}}} \right) \left( \frac{\mu_K(\mathbf{n})^2 \tau_K^*(\mathbf{n}_{t-1}; \psi_K)^{\frac{1}{m-1}}}{\mathfrak{N}\mathbf{n}^{\frac{1+\sigma}{m-1}} (\log \mathfrak{N}\mathbf{p}^+(\mathbf{n}_k))^{\frac{1}{m}}} \right).$$

Assuming that  $t'$  is large enough so that  $m > 2$  we can gain the succeeding bound via a use of Hölder's inequality with exponents  $\frac{m-1}{m-2}, m-1$ ,

$$\ll L_{t-1,m}^*(\sigma)^{\frac{m-2}{m-1}} \mathcal{D}_{t-1,m}(\sigma)^{\frac{1}{m-1}},$$

where for  $t, m$  positive integers and  $\hat{\sigma} \in (0, \frac{1}{4})$  we have defined

$$\mathcal{D}_{t-1,m}(\hat{\sigma}) := \sum_{\substack{\mathbf{n} \in \mathcal{P}_K \\ \omega_K(\mathbf{n}) \geq t-1}} \frac{\mu_K(\mathbf{n})^2 \tau_K^*(\mathbf{n}_k; \psi)}{\mathfrak{N}\mathbf{n}^{1+\hat{\sigma}} (\log \mathfrak{N}\mathbf{p}^+(\mathbf{n}_{t-1}))^{\frac{m-1}{m}}}.$$

To estimate  $\mathcal{D}_{t-1,m}(\hat{\sigma})$  we shall need the following lemma.

**Lemma 4.5.** *For  $\vartheta, \Gamma \in (0, \infty)$  define*

$$S(\Gamma; \vartheta) := \sum_{\substack{\mathfrak{N}\mathbf{p} \leq \Gamma \\ \mathbf{p} \in \mathcal{P}_K}} \frac{|1 + \psi_K(\mathbf{p}) \mathfrak{N}\mathbf{p}^{i\vartheta}|^2}{\mathfrak{N}\mathbf{p}}.$$

*There exists a constant  $B = B(K, \psi_K)$  such that the following holds uniformly in  $\vartheta$ ,*

$$S(\Gamma; \vartheta) \leq \begin{cases} 2 \log(1 + |\vartheta| \log \Gamma) + 2 \log \left( \frac{\log \Gamma}{1 + |\vartheta| \log \Gamma} \right) + O(1), & \text{if } 0 < |\vartheta| \leq 1 \\ 2 \log \log \Gamma + B \log \log(2 + |\vartheta|), & \text{otherwise,} \end{cases}$$

*where the implied constants are independent of  $\Gamma$  and  $\vartheta$ .*

*Proof.* For  $j = 1, -1$  we let

$$S_j(\Gamma; \vartheta) := \sum_{\substack{\mathfrak{N}\mathbf{p} \leq \Gamma \\ \mathbf{p} \in \mathcal{P}_K, \psi_K(\mathbf{p}) = j}} \frac{|1 + \psi_K(\mathbf{p}) \mathfrak{N}\mathbf{p}^{i\vartheta}|^2}{\mathfrak{N}\mathbf{p}}$$

so that  $S(\Gamma; \vartheta) = \sum_{j \in \{1, -1\}} S_j(\Gamma; \vartheta) + O(1)$ . Introduce the functions  $g_j : \mathbb{N} \rightarrow \mathbb{Z}_{\geq 0}$  via

$$g_j(n) := \#\{\mathbf{a} \in \mathcal{P}_K : \mathfrak{N}\mathbf{a} = n, \psi_K(\mathbf{a}) = j\}$$

and note that the condition  $\mathbf{p} \in \mathcal{P}_K$  forces  $\mathfrak{N}\mathbf{p}$  to be a rational prime. The quantitative version of Chebotarev's theorem provides positive constants  $c', \eta'$  such that

$$\sum_{p \leq \Gamma} g_j(p) = \sum_{\substack{\mathfrak{N}\mathbf{p} \leq \Gamma \\ \mathbf{p} \in \mathcal{P}_K, \psi_K(\mathbf{p}) = j}} 1 = \frac{\text{li}(\Gamma)}{2} + O(\Gamma e^{-c'(\log \Gamma)^{\eta'}}),$$

due to the standard bound

$$\sum_{\substack{\mathfrak{N}\mathbf{p} \leq \Gamma \\ f_{\mathbf{p}} > 1}} 1 \ll_{\varepsilon, K} \Gamma^{\frac{1}{2} + \varepsilon},$$

valid for all  $\varepsilon > 0$ . Hence, in the notation of [Ten15, Lem.III.4.13], we can use  $h(r) := |1 + e^{ir}|^2$  and directly modify its proof to show that for each  $w < \Gamma$  the following equality holds uniformly in  $\vartheta \neq 0$ ,

$$\sum_{\substack{w < \mathfrak{N}\mathfrak{p} \leq \Gamma \\ \mathfrak{p} \in \mathcal{P}_K \\ \psi_K(\mathfrak{p}) = j}} \frac{|1 + j\mathfrak{N}\mathfrak{p}^{i\vartheta}|^2}{\mathfrak{N}\mathfrak{p}} = \sum_{w < p \leq \Gamma} |1 + jp^{i\vartheta}|^2 \frac{g_j(p)}{p} = \log \left( \frac{\log \Gamma}{\log w} \right) + O \left( \frac{1}{|\vartheta| \log w} + \frac{1 + |\vartheta|}{e^{c''(\log w)^{\eta'}}} \right),$$

owing to  $\bar{h} = 2$  for our choice of  $h$ . This equality is parallel to [dlBT12, Eq.(3.16)], our proof can thus be concluded as the one of [dlBT12, Lem.2.5] by using it for suitable parameters  $w$  according to the value of  $\vartheta$  in relation to  $\Gamma$ .  $\square$

**Lemma 4.6.** *For all  $t, m$  as in Proposition 4.2 and  $\hat{\sigma} \in \mathbb{R} \cap (0, \frac{1}{4})$ , we have*

$$\mathcal{D}_{t-1,m}(\hat{\sigma}) \ll \frac{t-1}{\hat{\sigma} \left(1 - \frac{1}{2m}\right)^{t-1}}.$$

*Proof.* Fix an element  $\mathfrak{m} \in \mathcal{P}_K$ . The integral ideals  $\mathfrak{n}$  in  $\mathcal{D}_{t-1,m}(\hat{\sigma})$  with  $\mathfrak{n}_{t-1} = \mathfrak{m}$  are of the shape  $\mathfrak{n} = \mathfrak{m}\mathfrak{d}$ , where  $\mathfrak{d}$  is square-free and furthermore if  $\mathfrak{p}_i | \mathfrak{d}$  then  $i > i^+(\mathfrak{m})$ . Hence,

$$\mathcal{D}_{t-1,m}(\hat{\sigma}) \leq \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m}) = t-1}} \frac{\mu_K(\mathfrak{m})^2 \tau_K^*(\mathfrak{m}; \psi_K)}{\mathfrak{N}\mathfrak{m}^{1+\hat{\sigma}} (\log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}))^{\frac{m-1}{m}}} \prod_{i > i^+(\mathfrak{m})} \left( 1 + \frac{1}{\mathfrak{N}\mathfrak{p}_i^{1+\hat{\sigma}}} \right).$$

The last product is

$$\ll \zeta_K(1 + \hat{\sigma}) \exp \left( - \sum_{i \leq i^+(\mathfrak{m})} \mathfrak{N}\mathfrak{p}_i^{-1-\hat{\sigma}} \right) \ll \hat{\sigma}^{-1} \exp \left( - \sum_{i \leq i^+(\mathfrak{m})} \mathfrak{N}\mathfrak{p}_i^{-1-\hat{\sigma}} \right).$$

The inequality  $1 - \mathfrak{N}\mathfrak{p}_i^{-\hat{\sigma}} \leq \hat{\sigma} \log \mathfrak{N}\mathfrak{p}_i$  reveals that

$$\sum_{i \leq i^+(\mathfrak{m})} \mathfrak{N}\mathfrak{p}_i^{-1-\hat{\sigma}} \geq \sum_{i \leq i^+(\mathfrak{m})} \frac{1}{\mathfrak{N}\mathfrak{p}_i} - \hat{\sigma} \sum_{i \leq i^+(\mathfrak{m})} \frac{\log \mathfrak{N}\mathfrak{p}_i}{\mathfrak{N}\mathfrak{p}_i},$$

which, by the prime number theorem for  $K$ , is  $\ll \log \log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}) - \hat{\sigma} \log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m})$ . We can therefore bound  $\mathcal{D}_{t-1,m}(\hat{\sigma})$  by

$$\ll \frac{1}{\hat{\sigma}} \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m}) = t-1}} \frac{\mu_K(\mathfrak{m})^2 \tau_K^*(\mathfrak{m}; \psi_K)}{\mathfrak{N}\mathfrak{m} (\log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}))^{\frac{2m-1}{m}}} \frac{\mathfrak{N}\mathfrak{p}^+(\mathfrak{m})^{\hat{\sigma}}}{\mathfrak{N}\mathfrak{m}^{\hat{\sigma}}}$$

and, letting

$$T := \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m}) = t-1}} \frac{\mu_K(\mathfrak{m})^2 \tau_K^*(\mathfrak{m}; \psi_K)}{\mathfrak{N}\mathfrak{m} (\log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}))^{\frac{2m-1}{m}}},$$

allows us to deploy the inequality  $\mathfrak{N}\mathfrak{p}^+(\mathfrak{m}) \leq \mathfrak{N}\mathfrak{m}$  to infer that  $\mathcal{D}_{t-1,m}(\hat{\sigma}) \ll T/\hat{\sigma}$ . For  $\vartheta \in \mathbb{R}$  let

$$T(\vartheta) := \sum_{\substack{\mathfrak{m} \in \mathcal{P}_K \\ \omega_K(\mathfrak{m}) = t-1}} \frac{\mu_K(\mathfrak{m})^2 |\tau_K^*(\mathfrak{a}; \psi_K; \vartheta)|^2}{\mathfrak{N}\mathfrak{m} (\log \mathfrak{N}\mathfrak{p}^+(\mathfrak{m}))^{\frac{2m-1}{m}}}.$$

Note that alluding to (4.16) and using  $\tau^*(\mathfrak{m}, \psi; -\vartheta) = \overline{\tau^*(\mathfrak{m}, \psi; \vartheta)}$  provides us with

$$T \ll \int_0^\infty T(\vartheta)(1 + \vartheta^2)^{-1} d\vartheta.$$

Denote  $\mathfrak{p}' = \mathfrak{p}^+(\mathfrak{m})$ . Each ideal  $\mathfrak{m}$  in  $T(\vartheta)$  is the product of  $\mathfrak{p}'$  and  $k-1$  different prime ideals  $\mathfrak{p}_i \in \mathcal{P}_K$  that satisfy  $i < i^+(\mathfrak{m})$ . Therefore

$$T(\vartheta) \ll \frac{1}{(t-2)!} \sum_{\mathfrak{p}' \in \mathcal{P}_K} \frac{|\tau_K^*(\mathfrak{p}'; \psi; \vartheta)|^2}{\mathfrak{N}\mathfrak{p}'(\log \mathfrak{N}\mathfrak{p}')^{\frac{2m-1}{m}}} \left( \sum_{\substack{\mathfrak{N}\mathfrak{p} \leq \mathfrak{N}\mathfrak{p}' \\ \mathfrak{p} \in \mathcal{P}_K}} \frac{|\tau_K^*(\mathfrak{p}; \psi_K; \vartheta)|^2}{\mathfrak{N}\mathfrak{p}} \right)^{t-2}$$

and using Lemma 4.5 allows us to follow the arguments proving [dlBT12, Eq.(2.25),(2.26)] to acquire the bound

$$\int_0^1 T(\vartheta) d\vartheta \ll \frac{2^t}{(t-2)!} \left\{ \frac{(t-2)!}{(\frac{2m-1}{m} + 1)^{t-1}} + \frac{(t-1)!}{(\frac{2m-1}{m})^{t-1}} + \frac{(t-1)!}{(\frac{2m-1}{m} + 1)^{t-1}} \right\} \ll \frac{t-1}{(1 - \frac{1}{2m})^{t-1}}.$$

In the remaining range,  $\vartheta > 1$ , one can conjure up Lemma 4.5 and the proof of [dlBT12, Eq.(2.27)] to deduce the estimate

$$T(\vartheta) \ll \left(1 - \frac{1}{2m}\right)^{-(t-1)} \{\log(2 + \vartheta)\}^{\frac{B}{1 - \frac{1}{2m}}},$$

which, in light of

$$\int_1^\infty \{\log(2 + \vartheta)\}^{\frac{B}{1 - \frac{1}{2m}}} \vartheta^{-2} d\vartheta \ll_B 1,$$

is sufficient for our lemma.  $\square$

Assorting all appropriate estimates obtained so far validates

$$L_{t,m}^*(\sigma) - 4^{\frac{1}{m}} L_{t-1,m}^*(\sigma) \ll \frac{L_{t-1,q}^*(\sigma)^{\frac{m-2}{m-1}} (t-1)^{\frac{1}{m-1}}}{\sigma^{\frac{1}{m-1}} \left(1 - \frac{1}{2m}\right)^{\frac{t-1}{m-1}}} + \frac{1}{\sigma(t!)^{2/3}}. \quad (4.20)$$

Bringing into play the entity

$$L_{k,q}^*(\sigma) := L_{k,q}(\sigma) + \frac{4^{\frac{k}{q}}}{\sigma}$$

and noting that

$$L_{k+1,q}(\sigma) - 4^{\frac{1}{q}} L_{k,q}(\sigma) = L_{k+1,q}^*(\sigma) - 4^{\frac{1}{q}} L_{k,q}^*(\sigma), \frac{4^{\frac{t-1}{m}}}{\sigma} \leq L_{t-1,m}^*(\sigma)$$

allows to gain via (4.20) the following inequality,

$$L_{t,m}^*(\sigma) \leq L_{t-1,m}^*(\sigma) \left( 4^{\frac{1}{m}} + \frac{(t-1)^{\frac{1}{m-1}}}{4^{\frac{t}{m(m-1)}} \left(1 - \frac{1}{2m}\right)^{\frac{t-1}{m-1}}} + \frac{1}{4^{\frac{t-1}{m}} (t!)^{2/3}} \right).$$

Using the fact  $m = f(t) + O(1)$  shows that the middle term in the parenthesis is

$$\frac{1}{t^{\log 4 - \frac{1}{2} + o(1)}}.$$

Hence, there exists  $c_3 > 0$ , depending at most on  $K$  and  $\psi_K$  such that

$$L_{t,m}^*(\sigma) \leq e^{\frac{c_3}{m}} L_{t-1,m}^*(\sigma),$$

an estimate that concludes the proof of Proposition 4.2.

## REFERENCES

- [Brü12] J. Brüder, *Daniel's twists of Hooley's delta function*, Contributions in analytic and algebraic number theory, Springer Proc. Math., vol. 9, Springer, New York, 2012, pp. 31–82.
- [BS16] T. Browning and E. Sofos, *Counting rational points on quartic del Pezzo surfaces with a rational conic*, Submitted (2016).
- [dlBT12] R. de la Bretèche and G. Tenenbaum, *Oscillations localisées sur les diviseurs*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 669–693.
- [dlBT13] ———, *Sur la conjecture de Manin pour certaines surfaces de Châtelet*, J. Inst. Math. Jussieu **12** (2013), no. 4, 759–819.
- [FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), no. 2, 421–435.
- [Hoo79] C. Hooley, *On a new technique and its applications to the theory of numbers*, Proc. London Math. Soc. (3) **38** (1979), no. 1, 115–151.
- [HT86] R. R. Hall and G. Tenenbaum, *The average orders of Hooley's  $\Delta_r$ -functions. II*, Compositio Math. **60** (1986), no. 2, 163–186.
- [HT88] ———, *Divisors*, Cambridge Tracts in Mathematics, vol. 90, Cambridge University Press, Cambridge, 1988.
- [Shi80] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. reine angew. Math. **313** (1980), 161–170.
- [Ten85] G. Tenenbaum, *Sur la concentration moyenne des diviseurs*, Comment. Math. Helv. **60** (1985), no. 3, 411–428.
- [Ten86] ———, *Fonctions  $\Delta$  de Hooley et applications*, Séminaire de théorie des nombres, Paris 1984–85, Progr. Math., vol. 63, Birkhäuser Boston, Boston, MA, 1986, pp. 225–239.
- [Ten90] ———, *Sur une question d'Erdős et Schinzel. II*, Invent. Math. **99** (1990), no. 1, 215–224.
- [Ten15] ———, *Introduction to analytic and probabilistic number theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015.
- [Vau85] R. C. Vaughan, *Sur le problème de Waring pour les cubes*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 6, 253–255.
- [Vau86] ———, *On Waring's problem for cubes*, J. reine angew. Math. **365** (1986), 122–170.
- [Wie33] N. Wiener, *The Fourier integral and certain of its applications*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1933.

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